Solutions of Physics Brawl Online 2022

Problem 1 ... my hand is slipping off

3 points

According to information from Wikipedia, the longest escalators in Prague are at Náměstí Míru. They travel a distance $l = 87.1\,\mathrm{m}$ in $t = 140\,\mathrm{s}$. It is also said that the handrails on the escalators move $\delta v = 2.0\,\%$ faster than the stairs in order to ensure alertness. How much would your arm move relative to the rest of your body (if it was long enough) during the entire escalator ride, if you stood on the same stair while holding one spot the whole time?

Karel was wondering about escalators again.

The handrails are moving at a speed

$$u = v \cdot (1 + \delta v) \,,$$

so the hand will get to the end of the escalator ride in time

$$\tau = \frac{l}{u} = \frac{l}{v} \cdot \frac{1}{1 + \delta v} \,.$$

Meanwhile, the rest of the body travels the distance

$$l_0 = v \cdot \tau = l_0 \cdot \frac{1}{1 + \delta v} \,,$$

from which we find that the hand has moved relative to the rest of the body by

$$\Delta l = l - l_0 = \frac{\delta v}{1 + \delta v} \cdot l \,,$$

$$\Delta l = 1.71\,\mathrm{m} \,.$$

The hand would move by 1.71 m with respect to the legs. Therefore, it is not realistic for most people to hold themselves in one place all the time. It depends on how exactly the escalators are set up. However, the fact that you sometimes have to change the position of your hand (and you probably do so subconsciously) is usually not a fault of the escalators, but instead a desired feature.

Jiří Kohl
`jiri.kohl@fykos.org`

Karel Kolář
`karel@fykos.org`

Problem 2 ... comparison of slings

3 points

Lego got bored of bows, so he went to buy a sling. We approximate it as a spring with constant stiffness and a zero rest length. Lego can stretch the sling with force $F = 31.4\,\mathrm{N}$, and he wants it to store as much energy as possible while it is stretched. There were two slings at the shop, with stiffnesses $k_1 = 159\,\mathrm{N/m}$ and $k_2 = 265\,\mathrm{N/m}$. What will be the difference between the potential energies of those slings stretched with force F? Give a positive result if the sling with k_1 has more energy and a negative one if the opposite is true.

Lego got bored of bows.

The potential energy of a spring is given by $E_p = \frac{1}{2}ky^2$, where k is a stiffness of a spring, and y is its elongation. If we stretch the spring with the stiffness k_1 using the force F, its length will be increased by $y_1 = \frac{F}{k_1}$. Then for the potential energy of the first spring, we get

$$E_{p1} = \frac{1}{2} k_1 y_1^2 = \frac{F^2}{2k_1} \,.$$

Analogically we can express the potential energy of the second spring with the same elongation. Then, we calculate the difference between those two energies, which will give us

$$E_{p1} - E_{p2} = \frac{F^2}{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right) = 1.24\,\mathrm{J}\,,$$

which is our solution.

As we can see, if we fixate the force with which we are stretching the string, we will end up doing more work when the spring has smaller stiffness. Why don't we make all slings and bows with the least possible stiffness? The short answer would be that even if we don't consider the maximum force we can use, our arms are limited by the maximum length they can be apart. (More complex answer includes that force, which we can apply, isn't independent of the relative position of our hands...)

Šimon Pajger
`legolas@fykos.org`

Problem 3 ... mountain biking 3 points

Matěj is riding on a mountain bike trail, passing oncoming cyclists. At the top of the hill, exhausted riders are moving at an average speed of $5\,\mathrm{km{\cdot}h^{-1}}$ in both directions and Matěj meets them here at an average rate of 0.02 per second. On the other hand, at the lowest point of the trail, the cyclists are revved up (from both directions) and moving at an average speed of $50\,\mathrm{km{\cdot}h^{-1}}$. How many cyclists does Matěj meet here on average? The bike trail has no turnoffs, and Matěj always moves at the speed of an average cyclist. *Matěj mountain biked.*

The problem is solvable using a trick and thus has a trivial solution. Assuming that no one has crashed, to conserve the number of cyclists, the same number of riders must pass each point on the trail per second. Due to Matěj moving at the same speed as the oncoming cyclists, the number of riders encountered per second is double (relative to a static observer) but remains constant. The solution is therefore 0.02 cyclists per second.

Matěj Mezera
`m.mezera@fykos.org`

Problem 4 ... hunter, shoot yourself 3 points

What is the largest radius a planet can have for a person to be able to shoot oneself on it? Hunter shoots just above its surface, parallel to it, and the planet will have no atmosphere. The velocity of a flying bullet is $v = 380\,\mathrm{m{\cdot}s^{-1}}$, and the density of the planet is $\rho = 3\,200\,\mathrm{kg{\cdot}m^{-3}}$. Furthermore, assume the planet to be spherical and homogeneous.

Karel wanted planetary shooting.

The problem statement basically asks us what the radius of the planet must be, for the bullet to move in a circular path just above its surface (it would move in an elliptical orbit with greater velocity). Denoting m the mass of the bullet, M the mass of the planet, and R its radius, we are interested with the equality of forces

$$m\frac{v^2}{R} = G\frac{mM}{R^2}\,,$$

where the mass M can be further expressed as

$$M = \rho V = \frac{4}{3}\pi\rho R^3 \,.$$

By addition and rearranging the equation, we get

$$R = v\sqrt{\frac{3}{4\pi\rho G}} \doteq 402\,\mathrm{km}\,.$$

Vojtěch David
`vojtech.david@fykos.org`

Problem 5 ... falling icicles 3 points

A small icicle broke loose from the edge of the roof and started its free fall along the vertical wall of the house. The icicle flew past a window of height $h = 1.50\,\mathrm{m}$ during a time $t = 0.10\,\mathrm{s}$. What is the distance between the edge of the roof and the top of the window?

Kuba was once almost hit by an icicle.

We know that the height of the window is h and that the icicle was falling past it during the time t, so it has traveled the path h in the time t. Clearly, the icicle already had some non-zero velocity at the top edge of the window. We shall therefore use the formula for the path of a uniformly accelerated rectilinear motion with a non-zero initial speed

$$h = v_0 t + \frac{1}{2}gt^2 \,.$$

The only quantity in this relation that we do not know is the speed at the upper edge of the window v_0. We can therefore express it as

$$v_0 = \frac{h - \frac{1}{2}gt^2}{t} \,.$$

Finally, it remains to realize that if the icicle started its fall from rest and it gained the speed v_0 in a time $t_0 = v_0/g$. Thus, from the moment it started falling to the moment it was at the top edge of the window, it fell by

$$h_0 = \frac{1}{2}gt_0^2 = \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = \frac{\left(h - \frac{1}{2}gt^2\right)^2}{2gt^2} \doteq 10.7\,\mathrm{m}\,.$$

Šimon Pajger
`legolas@fykos.org`

Problem 6 ... worn tire

3 points

The tires on our new car have a diameter of $D = 634.5\,\mathrm{mm}$ and a tread depth of $d_0 = 8.0\,\mathrm{mm}$. The technician set the car's speedometer to measure accurately for this original wheel size. However, we drive frequently, so after a while, the tire tread came down to the minimum depth for summer use, $d_{\mathrm{min}} = 1.6\,\mathrm{mm}$. How does the actual speed now differ from the speedometer reading? Give the answer as the ratio of the speed on the speedometer to the actual speed.

Karel was refueling.

We consider the wheel to be perfectly circular; then its circumference is $o_0 = \pi D$. After the tire is worn out, we reach the diameter $D_1 = D - 2(d_0 + d_{min})$ and the new circumference of the wheel is $o_1 = \pi\left(D - 2d_0 + 2d_{\mathrm{min}}\right)$. When the car travels the circumference of the new wheel, the "speedometer thinks" it has traveled a little more, namely the length of the original circumference. The ratio of speeds shown by the speedometer will correspond to the ratio of the circumferences, i.e.

$$k = \frac{o_0}{o_1} = \frac{\pi D}{\pi\left(D - 2d_0 + 2d_{\mathrm{min}}\right)} = \frac{D}{D - 2d_0 + 2d_{\mathrm{min}}} \doteq 1.02\,.$$

The speedometer then shows $102\,\%$ of the actual speed.

In fact, the speedometer should always be set to show a little more than the actual speed, even with unworn tread. The real situation is also complicated by the fact, that the tires change shape during the motion; we're not driving just on a flat road, etc. However, these effects are relatively small.

Karel Kolář
`karel@fykos.org`

Problem 7 ... Drop Da Bomb

3 points

At the end of The Simpsons Movie, Homer and Bart are riding a motorcycle on the inside of a glass dome. Assume the dome has a radius of $1.00\,\mathrm{km}$. What minimum speed does the motorcycle have to travel at to ensure that Homer and Bart do not fall off even at the highest point of the dome? *Jindra was testing his new motorcycle at The Ondřejov Observatory.*

We will be using a non-inertial system coupled to the motorcycle. Two forces act on the motorcycle at the top of the dome: a downward gravitational force and an upward centrifugal force. The limiting case occurs when the two forces are equal

$$mg = \frac{mv^2}{r}.$$

We have denoted the radius of the sphere by $r = 1.00\,\mathrm{km}$, m is the mass of the motorcycle with riders, $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$ is the acceleration due to gravity, and v is the minimum speed of the motorcycle. Expressed

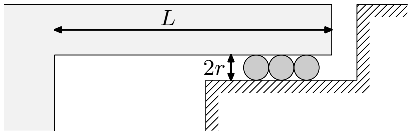
$$v = \sqrt{gr} = 99.0\,\mathrm{m{\cdot}s^{-1}} = 357\,\mathrm{km{\cdot}h^{-1}}.$$

Homer and Bart would have to travel at $357\,\mathrm{km{\cdot}h^{-1}}$.

Jindřich Jelínek
`jjelinek@fykos.org`

Problem 8 ... moving bridge

3 points



A road bridge of length $L = 8\,\mathrm{m}$ is firmly anchored on one side. The other side is placed on sturdy cylinders of radius $r = 6\,\mathrm{cm}$. The cylinders lay on a rigid flat surface on which they can move freely in a horizontal direction. By what angle will the cylinders rotate between winter, when the temperature is $t_z = -10\,\mathrm{^\circ C}$, and summer, when the temperature is $t_l = 30\,\mathrm{^\circ C}$? The coefficient of linear thermal expansion of the bridge is $\alpha = 1.0 \cdot 10^{-5}\,\mathrm{K^{-1}}$. Note that the cylinders do not slip and that there is a gap between the road and the moving end of the bridge.

Jarda can feel the ground moving beneath his feet.

The length of the bridge changes by $\Delta l = l\alpha\Delta T$, where l is its original length, α is the coefficient of linear thermal expansion, and $\Delta T = 40\,\mathrm{^\circ C}$ is the temperature difference between winter and summer.

The moving end of the bridge is therefore displaced by $\Delta l = 3.2\,\mathrm{mm}$. As the end of the bridge moves, the position of the cylinders changes. Since the cylinders do not slip, the distances on both flat and curved surfaces are the same. When the end of the bridge moves, the cylinder moves half the distance, because it does not slip at the bottom or the top - the center of the cylinder moves forward at a certain velocity, but the top of the cylinder moves faster by a factor of the angular velocity times the radius of the cylinder.

Thus the angle in degrees is calculated as

$$\varphi = \frac{l\alpha\Delta T}{2}\frac{360\,^\circ}{2\pi r} = 1.5^\circ\,,$$

where r is the radius of the cylinder.

Jaroslav Herman
`jardah@fykos.org`

Problem 9 ... two rectangular cuboids on a hill

4 points

Let us have "a hill" made of two inclined planes with inclinations α_1 and α_2 with respect to the horizontal plane. We lay a rectangular cuboid with the mass $m_1 = 10\,\mathrm{kg}$ on the first plane with inclination $\alpha_1 = 25^\circ$. On the second plane with inclination $\alpha_2 = 35^\circ$, we place a different rectangular cuboid with the mass $m_2 = 15\,\mathrm{kg}$. Then we connect the cuboids with a massless rope that is led through a massless pulley at the top of the hill. The rope is always stretched parallel to the given plane, the coefficient of dynamic friction between the cuboids and inclined planes is $f = 0.15$, and the coefficient of static friction is small enough for cuboids to start moving. What is the acceleration of the cuboid m_1 if both cuboids are at rest at the beginning? The positive sign will indicate acceleration downhill, and a negative sign will indicate acceleration uphill.

Lego was hiking.

The attentive solver will immediately notice that the rectangular cuboid m_2 is heavier and on the inclined plane with a greater slope. So, intuitively, the cuboid m_2 will accelerate downhill and m_1 will accelerate uphill. We will verify this by calculation. It is important to note, that if we made this calculation with the assumption that m_1 is accelerating downhill, we would get negative acceleration. However, this acceleration is not our solution, because it was calculated with the wrong assumption, that the cuboids are moving with the opposite acceleration to what they truly are. Hence, the friction forces have the opposite orientation (this incorrect solution is therefore larger in absolute value than the correct one).

Enough talking about how not to solve this problem; let's see how to solve it correctly. The cuboid m_2 is pulled down by a component of its weight parallel with the inclined plane, its magnitude, therefore, is $F_{\mathrm{k}2} = m_2 g \sin\alpha_2$. It will be slowed down by the component of its weight perpendicular to the base of the cuboid, i.e. with magnitude $F_{\mathrm{t}2} = f m_2 g \cos\alpha_2$. Moreover, it will be pulled upwards by the rope with a force whose magnitude T we do not know. The equation of motion for this cuboid will be $m_2 a_2 = F_{\mathrm{k}2} - F_{\mathrm{t}2} - T$. The forces for the cuboid m_1 will look similar, except that the rope will pull it up in the direction of its acceleration, while its weight will pull it down, i.e. in the opposite direction of its acceleration. The equation of motion for it will therefore be $m_1 a_1 = -F_{\mathrm{k}1} - F_{\mathrm{t}1} + T$.

Since the rope and the pulley are intangible, the rope pulls both cuboids with a force of the same magnitude T. At the same time, the acceleration of the cuboids must be equally large, so in our equations, $a_1 = a_2$. What remains is to sum up the equations to get rid of the unknown T and express a_2:

$$
\begin{aligned}
(m_2 + m_1) a_2 &= F_{\mathrm{k}2} - F_{\mathrm{t}2} - F_{\mathrm{k}1} - F_{\mathrm{t}1}\,,\\
a_2 &= g\frac{m_2 \sin\alpha_2 - f m_2 \cos\alpha_2 - m_1 \sin\alpha_1 - f m_1 \cos\alpha_1}{m_2 + m_1}\,,\\
a_2 &= 0.46\,\mathrm{m{\cdot}s^{-2}}\,.
\end{aligned}
$$

Let us note that the problem statement asked for a negative acceleration if the cuboid m_1 accelerates uphill, so the correct answer is $-0.46\,\mathrm{m{\cdot}s^{-2}}$.

Šimon Pajger
legolas@fykos.org

Problem 10 ... energy-saving electric kettle — 4 points

When prices rise, you must save. What percentage of our finances will we save if we heat less water for tea and more economically at the same time? The original method was to heat $V_1 = 1.00\,\mathrm{l}$ of water from the original temperature $t_0 = 18.5\,\mathrm{^\circ C}$ (both the kettle and water temperature) to the boiling point $t_\mathrm{v} = 98.5\,\mathrm{^\circ C}$. We use a kettle with heat capacity $C = 442\,\mathrm{J{\cdot}K^{-1}}$. The new method will be different in a smaller volume of heated water $V_2 = 0.800\,\mathrm{l}$ and in a lower reached temperature of $t_2 = 83\,\mathrm{^\circ C}$. Regardless of the current price development, consider the price of electricity for consumers as constant $p = 4.45\,\mathrm{Kč{\cdot}kWh^{-1}}$. Assume that the efficiency of the kettle is constant $95\,\%$ and that the temperature is the same in the whole volume at any given moment. *Karel was looking at the rising inflation.*

This is a relatively straightforward application of the calorimetric equation, which is only made more complicated by the fact that we are comparing two situations and at the end, we will have to calculate their ratio. The price given in the problem statement isn't necessary to solve the problem. The question is aimed at the percentage saving, which does not depend on the price of electricity if it is constant. However, if we heated in the first way before the price increase and in the second way after the price increase, we would need to know the current prices. Similarly, the information about efficiency is superfluous as well, because it only changes the absolute values of energy/heat, but the relative savings remain the same.

The heat received in the first case is the sum of the heat received by the water and by the kettle. Thus, the total

$$Q_1 = m_1 c\Delta t_1 + C\Delta t_1 = (\rho V_1 c + C)(t_v - t_0) ,$$

where $m_1 = \rho V_1$ is the mass of water, ρ is the density of water, c is the specific heat capacity of water. We assume that the material constants for water are sufficiently accurate to be constant for the given temperature range. Similarly, we can write a relation for the more economical heat

$$Q_2 = m_2 c\Delta t_2 + C\Delta t_2 = (\rho V_2 c + C)(t_2 - t_0) ,$$

where the variables are denoted analogously.

We want to compare the percentage savings with the original consumption. Let's denote the savings ratio as k. As mentioned earlier, in our case, the savings ratio is the same as the heat ratio

$$k = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1} = 1 - \frac{\rho V_2 c + C}{\rho V_1 c + C}\frac{t_2 - t_0}{t_\mathrm{v} - t_0} \doteq 34.0\,\%$$

From the final expression, we see that if we were satisfied with the result neglecting the heat capacity of the kettle, we would not even need to know the density of water nor the specific heat capacity of water. However, in that case, we would obtain the result 35.5 %, which is noticeably different within the specified number of significant digits, so we did not recognize the result with the neglected heat capacity of the kettle as valid.

Karel Kolář
karel@fykos.org

Problem 11 ... induced voltage reloaded

3 points

Let us have an area of a homogenous magnetic field. The area has a rectangular cross-section with sides $a = 3.0\,\mathrm{m}$, $b = 2.0\,\mathrm{m}$, and the magnetic field vector $B = 1.0\cdot 10^{-3}\,\mathrm{T}$ is perpendicular to it. We place a sufficiently long wire parallel to the diagonal of the rectangle so that it all lies entirely outside of the area with the magnetic field. We connect the ends of the wire to a voltmeter.

Now we start moving the wire in a rectilinear motion at a speed of $v = 0.20\,\mathrm{m{\cdot}s^{-1}}$ perpendicularly to the direction of the magnetic field vector and the diagonal to which it is parallel. What is the peak magnitude of voltage displayed by the voltmeter during the movement of the wire?

Vojta was proofreading.

We will calculate the voltage induced on a conductor of length l moving with a speed v perpendicularly to the magnetic field of induction B using

$$U = Bvl .$$

Since we are interested in the highest value of the voltage, the length of the part of the conductor inside the magnetic field must be maximal, as all other parameters are constant. The length can be at most $\sqrt{a^2 + b^2}$, so we get

$$U_\mathrm{max} = Bv\sqrt{a^2 + b^2} \doteq 0.72\,\mathrm{mV} .$$

Vojtěch David
vojtech.david@fykos.org

Problem 12 ... Archimedeskugeln 5 points

Consider two identical containers of mass $M_\mathrm{n} = 340.0\,\mathrm{g}$ *with the identical amount of water of mass* $M_\mathrm{w} = 150.0\,\mathrm{g}$*. We place each of the containers on a scale separately. Into the first container, we immerse a steel ball with density* $\rho_\mathrm{o} = 7\,850\,\mathrm{kg{\cdot}m^{-3}}$*, which is suspended from an external frame that does not stand on the scale. The ball is fully immersed and does not touch the bottom. In the second container, we attach a ball made of polystyrene foam with density* $\rho_\mathrm{p} = 27.0\,\mathrm{kg{\cdot}m^{-3}}$ *and radius* $r_\mathrm{p} = 1.5\,\mathrm{cm}$ *to the bottom with a thin string. The ball is fully submerged, not breaking the surface. What must be the radius of the steel ball so the two scales will display the same weight?*

Jindra has been learning German to hide the balls in the title.

In the first case, the force of gravity $\rho_\mathrm{o} V_\mathrm{o} g$ pointing downwards acts on a steel ball of volume V_o. In the upward direction, the two forces act – the buoyancy $\rho_\mathrm{w} V_\mathrm{o} g$ and the tensile force of the string T_o. The water in the container is subject to downward force $\rho_\mathrm{w} V_\mathrm{o} g$, which is a reaction to the buoyant force acting on the ball. Thus, the first container pushes on the scale with a total force $M_\mathrm{n} g + M_\mathrm{w} g + \rho_\mathrm{w} V_\mathrm{o} g$, where the first two terms express the gravitational forces of the container and the water.

In the second case, the upward buoyant force of the water $\rho_\mathrm{w} V_\mathrm{p} g$ acts on the foam ball of volume V_p. In the downward direction, the gravitational force $\rho_\mathrm{p} V_\mathrm{p} g$ acts. From the equilibrium of forces acting on this ball, we determine the tensile force of the string as

$$T_\mathrm{p} = (\rho_\mathrm{w} - \rho_\mathrm{p})\, V_\mathrm{p} g\,,$$

which acts downwards on the foam ball.

The reaction force from the ball $\rho_\mathrm{w} V_\mathrm{p} g$ acts downwards on the water. However, the string also pulls on the bottom with a force of magnitude T_p upwards, which lifts the container. The total force acting on the scale is, therefore

$$M_\mathrm{n} g + M_\mathrm{w} g + \rho_\mathrm{w} V_\mathrm{p} g - (\rho_\mathrm{w} - \rho_\mathrm{p})\, V_\mathrm{p} g = M_\mathrm{n} g + M_\mathrm{w} g + \rho_\mathrm{p} V_\mathrm{p} g\,.$$

We could have derived the same result a little easier if we realized that the container, the water, and the polystyrene-foam ball form one system with mass $M_\mathrm{n} + M_\mathrm{w} + \rho_\mathrm{p} V_\mathrm{p}$, on which the gravitational force $(M_\mathrm{n} + M_\mathrm{w} + \rho_\mathrm{p} V_\mathrm{p})\, g$ acts.

Both scales display the same value if $\rho_\mathrm{w} V_\mathrm{o} = \rho_\mathrm{p} V_\mathrm{p}$ holds (the masses of water and container cancel out), from which we obtain

$$r_\mathrm{o} = r_\mathrm{p} \sqrt[3]{\frac{\rho_\mathrm{p}}{\rho_\mathrm{w}}} = 0.45\,\mathrm{cm}\,.$$

Interestingly, the desired radius does not depend on the density of the steel.

Jindřich Jelínek
`jjelinek@fykos.org`

Problem 13 ... neglecting the Earth's movement 4 points

Have you ever considered that when we calculate the free fall of a body on Earth, we neglect the Earth's motion towards the body? This calculation corresponds to an Earth with infinite mass. However, is it possible to measure the Earth's motion, at least in theory? According to *today's physics, we generally cannot measure distances smaller than Planck's length, which is* $l_\mathrm{P} = \sqrt{\hbar G/c^3} \doteq 1.616 \cdot 10^{-35}\,\mathrm{m}$*. What is the minimum mass that a body needs to have for the Earth to move from its initial position by at least* $n = 10^4$ *Planck's lengths from the point when this body is released until the point when it impacts the Earth? The body is released at the height* $h = 1.00\,\mathrm{m}$ *above the Earth's surface. Neglect the Earth's rotation, atmosphere, and any other motion of the Earth.* *Karel was thinking about approximations and Planck units.*

Solution using free fall

For simplicity, let's assume we're dealing with simple free fall. During some time, the body moves by h and meanwhile, the Earth moves by nl_P. We could also consider that the displacement of the body during free fall is actually smaller because the Earth moved towards it, but this difference is so small that we can neglect it.

Since we're dealing with motion from rest with constant acceleration in our reference frame,

$$h = \frac{1}{2}a\,t^2\,, \quad s_\oplus = \frac{1}{2}a_\oplus t^2\,,$$

where a is the acceleration of the body, t is the duration of the fall, $s_\oplus$ is the displacement of the Earth and $a_\oplus$ is the acceleration of the Earth. Both accelerations are gravitational, so

$$a = \frac{GM_\oplus}{R_\oplus^2}\,, \quad a_\oplus = \frac{Gm}{R_\oplus^2}\,,$$

where $R_\oplus$ is the radius of the Earth, and we can again neglect both h and nl_P compared to this radius and consider the acceleration during the fall to be constant.

The durations of the "falls" are equal, so

$$\frac{2h}{a} = \frac{2s_\oplus}{a_\oplus} \quad \Rightarrow \quad s_\oplus = \frac{a_\oplus}{a}h = \frac{m}{M_\oplus}h\,.$$

We need the displacement to be greater than nl_P, so

$$s_\oplus > nl_\mathrm{P} \quad \Rightarrow \quad nl_\mathrm{P} < \frac{m}{M_\oplus}h \quad \Rightarrow \quad m > nl_\mathrm{P}\frac{M_\oplus}{h} \doteq 9.65 \cdot 10^{-7}\,\mathrm{kg} = 0.965\,\mathrm{mg}\,.$$

The mass of the body would need to be greater than 0.965 mg.

Solution using center of mass

Since the problem statement says we're dealing with an isolated system of the Earth and the body (we don't consider other motions of the Earth), their common center of mass must perform uniform linear motion according to Newton's 1st law. If we're looking at the problem in the reference frame of this center of mass, we know that it does not move. If we denote the distances of the center of the Earth and the body from the common center of mass by $d_\oplus$ and d respectively, a well-known formula says that

$$d_\oplus M_\oplus = dm\,, \tag{1}$$

where $M_\oplus$ is the mass of the Earth. The height above the Earth's surface is $h + R_\oplus = d_\oplus + d$, where $R_\oplus$ is the radius of the Earth.[1] The formula for the distances of the centers of mass 1 holds all the time, during the fall and even after the body's impact on the surface. At the moment of impact, the distances change to $d'_\oplus$ and d', which satisfy $R_\oplus = d'_\oplus + d'$ and $d'_\oplus M_\oplus = d'm$. We're interested in displacements, not the distances themselves. Therefore, let's subtract these equations

$$\Delta d_\oplus M_\oplus = \Delta dm \,, \qquad h = \Delta d_\oplus + \Delta d \,.$$

Solving this system of two equations with two unknowns, we find the formula for displacement

$$\Delta d_\oplus = \frac{mh}{m + M_\oplus} \approx \frac{mh}{M_\oplus} \,.$$

We're looking for the smallest m which satisfies $\Delta d_\oplus > nl_\mathrm{P}$

$$m > \frac{M_\oplus n l_\mathrm{P}}{h} \doteq 9.65 \cdot 10^{-7}\,\mathrm{kg} = 0.965\,\mathrm{mg} \,.$$

It turns out that the mass of the body would need to be greater than approximately a milligram.

Finally, note that even 10^4 Planck lengths isn't a displacement which today's experimental physics would be able to measure for something as large as the Earth, so this problem is really just a theoretical plaything. Another alternative solution could use conservation of momentum.

Karel Kolář
karel@fykos.org

Jozef Lipták
liptak.j@fykos.org

Problem 14 ... hot light bulb

4 points

Danka has a lamp that has an old light bulb with an input power of $P = 60\,\mathrm{W}$ on her desk. At a distance $h = 40\,\mathrm{cm}$ below the bulb, there is a computer mouse with an elliptical cross-section. The semi-major axis of the ellipse is $a = 60\,\mathrm{mm}$ and the semi-minor $b = 30\,\mathrm{mm}$. When the light bulb was turned on, the mouse had a room temperature $T_1 = 23.0\,\mathrm{^\circ C}$. What will be the temperature of the mouse after 90 minutes of illumination? Assume that the bulb radiates isotropically into the entire space, $\eta = 83\%$ of energy hitting the mouses cross-section is converted into heat, while mouse is not losing heat itself. The heat capacity of the mouse is $C = 200\,\mathrm{J{\cdot}K^{-1}}$. *Danka's mouse is warming up from the lamp.*

From the calorimetry formula, we know that the heat Q received by the mouse is equal to

$$Q = C(T_2 - T_1) \,,$$

where T_2 is the temperature of the mouse after 90 minutes of illumination. This is equal to the heat it receives from the bulb. The bulb has a heat output power equal to $P\eta$, distributed evenly throughout the space. At distance h, the heat output power of the bulb per unit area is equal to $\frac{P\eta}{4\pi h^2}$. Multiplying this quantity by the area occupied by the cross-section of the mouse and by the time t gives the total heat that is supplied to the mouse. Since the mouse has non-zero dimensions, when its center is at a distance h from the bulb, its edges are at a slightly greater distance. However, we can calculate that the angle at which the longest dimension of

[1] Here we should correctly use the distance from the center of the Earth to the surface. Next, we silently assume in the whole solution that the body falls along the line connecting its initial position and the center of the Earth. However, this is a good approximation in our problem for a real non-spherical Earth.

the mouse is visible (as viewed from the bulb) is approximately 15 °. From this, we can calculate that the difference in the distances between the edge and center of the mouse from the bulb is negligible. We can then assume that the same energy flux is delivered to the entire surface of the mouse. The mouse's cross-section is the area of an ellipse, hence πab. From these observations, we obtain the equation for the equality of the radiated and received heat

$$\frac{P\eta t}{4\pi h^2}\pi ab = C(T_2 - T_1)\,.$$

We rearrange and express the search temperature T_2

$$T_2 = T_1 + \frac{P\eta tab}{4Ch^2}\,.$$

After plugging in the numerical values, we get $T_2 \doteq 27\,^\circ\mathrm{C}$.

Daniela Dupkalová
daniela@fykos.org

Problem 15 ... bead with spring 4 points

A bead of mass $m = 358\,\mathrm{g}$ is strung on a straight wire (so the bead can only move in one direction). We attach to the bead an ideal spring with zero rest length and spring constant $k = 979\,\mathrm{N{\cdot}m^{-1}}$. We fix the other end of this spring at a distance $l = 323\,\mathrm{mm}$ from the line on which the wire lies. What will be the period of the small oscillations of the bead around its equilibrium position? *Lego hasn't set a problem on small oscillation for a long time.*

Of course, we could solve the problem by calculating the force on the bead after a deflection by a small Δx, but it is quicker and more elegant to calculate how its potential energy increases after such a deflection.

The potential energy of a spring with zero rest length is equal to

$$E_p = \frac{1}{2}ky^2\,,$$

where y is its length. The equilibrium position of the bead is naturally at the point on the wire closest to the spring attachment point (because that is when the spring is shortest). From the problem statement, we know that the attachment point is at a distance l from the wire, so in the equilibrium position, the spring has a direction perpendicular to the wire and a length equal to l.

When we move the bead from its equilibrium position by Δx, we move it only perpendicular to the equilibrium direction of the spring. The new position of the spring will therefore be the hypotenuse of a right triangle, while the equilibrium position of the spring l and the deflection of the bead Δx are the legs. Then we can directly calculate the square of the new length of the spring as $l^2 + \Delta x^2$ from Pythagoras' theorem. The potential energy of the spring will therefore be

$$E_p = \frac{1}{2}k\left(l^2 + \Delta x^2\right)\,.$$

However, we must remember that the potential energy of the spring in the equilibrium position was $kl^2/2$, so by deflecting the bead by Δx, the potential energy increased by $k\Delta x^2/2$. Now we can either observe that this corresponds to a linear harmonic oscillator with stiffness k, or we can differentiate the potential energy by position and obtain that the force that counteracts towards the deflection Δx is $k\Delta x$. Either way, it remains to fit this to the formula for the period of a linear harmonic oscillator

$$T = 2\pi\sqrt{\frac{m}{k}} \doteq 0.120\,\mathrm{s}\,.$$

Interestingly, the bead oscillates with this period independently of the magnitude of its initial deflection on the wire; hence, we do not need to restrict to small oscillations.

Šimon Pajger
legolas@fykos.org

Problem 16 ... ballad of a sinful soul

5 points

Jarda's FYKOS soul will once be relieved of its physical burden and head to heaven to meet the FYKOS bird. However, weighed down by sins, it will slip into hell, where it will be enclosed in a cauldron of $V = 666\,\mathrm{cm^3}$. *It will remain preserved there at drastic conditions of* $T = 666\,^\circ\mathrm{C}$ *and* $p = 666 \cdot 10^5\,\mathrm{Pa}$, *while it will try to get rid of the sin molecules to rise again. The molar mass of sin is* $666\,\mathrm{g{\cdot}mol^{-1}}$. *How many sins will Jarda accumulate during his time at FYKOS? The mass of a pure soul is* $21\,\mathrm{g}$, *and its density at normal conditions is* $0.70\,\mathrm{kg{\cdot}m^{-3}}$. *Consider the soul and the sins to be an ideal gas.* *Jarda's proposition was not politically correct.*

Jarda's soul boiling in an isolated cauldron satisfies the ideal gas equation of state

$$\frac{pV}{T} = nR\,,$$

where R is the gas constant and $n = n_\mathrm{h} + n_\mathrm{č}$ is the sum of the number of sin and pure-soul moles.

Jarda's soul was originally as pure as any other, but sins found a way to get into it. We calculate the number of moles of the pure soul from its molar mass and $m = 21\,\mathrm{g}$ as

$$n_\mathrm{č} = \frac{m}{M_\mathrm{č}}\,,$$

where $M_\mathrm{č}$ is obtained from another equation of state

$$\frac{p_\mathrm{a}}{T_\mathrm{a}\rho} = \frac{R}{M_\mathrm{č}}\,,$$

which comes from the density information at normal conditions T_a and p_a. If we substitute, we get the number of moles of sins as

$$n_\mathrm{h} = \frac{pV}{TR} - n_\mathrm{č} = \frac{pV}{TR} - \frac{mp_\mathrm{a}}{RT_\mathrm{a}\rho}\,.$$

By multiplying by the molar mass, we get

$$m_\mathrm{h} = \frac{M_\mathrm{h}}{R}\left(\frac{pV}{T} - \frac{mp_\mathrm{a}}{T_\mathrm{a}\rho}\right) = 2.95\,\mathrm{kg}\,.$$

Amen.

Jaroslav Herman
jardah@fykos.org

Problem 17 ... is that what you meant, Mr. Planck? 5 points

Determine the Planck's surface tension.

Hint: Planck units are those formed by combining three fundamental physical constants – the gravitational constant G, the reduced Planck constant $\hbar$ and the speed of light c. Just for the record, the standard model of quantum physics fails on Planck scales in the microworld, and the effects of quantum gravity (of which we know nothing yet) must be taken into account.

Jindra came up with a problem that has a shorter statement (without a hint) than the title.

The three physical constants to determine the Planck scales are the gravitational constant $G = 6.673 \cdot 10^{-11}\,\mathrm{kg^{-1}{\cdot}m^3{\cdot}s^{-2}}$, the Planck constant $\hbar = 1.055 \cdot 10^{-34}\,\mathrm{kg{\cdot}m^2{\cdot}s^{-1}}$, and the speed of light $c = 2.998 \cdot 10^8\,\mathrm{m{\cdot}s^{-1}}$. The surface tension has the unit $\mathrm{N{\cdot}m^{-1}}$ or $\mathrm{kg{\cdot}s^{-2}}$.

We will use dimensional analysis. We assume that the Planck surface tension σ_P is the product of the powers of the three constants

$$\sigma_\mathrm{P} = CG^\alpha \hbar^\beta c^\gamma. \tag{2}$$

C is a dimensionless constant that we cannot determine from dimensional analysis. When deriving Planck units, $C = 1$ is the standard convention. However, when using dimensional analysis in general, it is better to leave the unknown C in the expression to make it clear that the correct relationship may differ by a multiple of the constant.

We plug the units of the physical quantities into the equation (2)

$$\mathrm{kg{\cdot}s^{-2}} = (\mathrm{kg^{-1}{\cdot}m^3{\cdot}s^{-2}})^\alpha (\mathrm{kg{\cdot}m^2{\cdot}s^{-1}})^\beta (\mathrm{m{\cdot}s^{-1}})^\gamma = \mathrm{kg}^{-\alpha+\beta} \cdot \mathrm{m}^{3\alpha+2\beta+\gamma} \cdot \mathrm{s}^{-2\alpha-\beta-\gamma}.$$

Since the number of units on the left and right sides must be the same, we get a system of three linear equations with three unknowns.

$$-\alpha + \beta = 1$$
$$3\alpha + 2\beta + \gamma = 0$$
$$-2\alpha - \beta - \gamma = -2$$

Summing the second and third equations gives $\alpha + \beta = -2$. Combined with the first equation, we get $\alpha = -3/2$, $\beta = -1/2$. This yields $\gamma = 11/2$. The Planck surface tension is

$$\sigma_\mathrm{P} = \sqrt{\frac{c^{11}}{\hbar G^3}} = 7.49 \cdot 10^{78}\,\mathrm{N{\cdot}m^{-1}}.$$

Jindřich Jelínek
`jjelinek@fykos.org`

Problem 18 ... red-hot resistance wire 5 points

Consider a wide resistive wire which at temperature $T_0 = 20\,^\circ\mathrm{C}$ has the length $l_0 = 10\,\mathrm{m}$ and resistance $R_0 = 1.23\,\Omega$. As the current passes through it, it heats up to $T = 100\,^\circ\mathrm{C}$, changing its dimensions and specific resistance. Our wire has a coefficient of linear thermal expansion $\alpha_l = 2.43 \cdot 10^{-3}\,\mathrm{K^{-1}}$ and its temperature coefficient of specific electrical resistance is $\alpha_R = 3.92 \cdot 10^{-3}\,\mathrm{K^{-1}}$. What will be the resistance of the wire at temperature T if it is connected *to a source such that its dimensions can change with temperature?*

Karel was learning to teach and thought about material constants.

Let us denote the specific electrical resistance of a wire at temperature T_0 as ρ_0 and its cross section as S_0. The standard relation that links the length of a body to its temperature through the coefficient of linear thermal expansion has the following form

$$l = l_0 \left(1 + \alpha_l \left(T - T_0\right)\right) .$$

A similar relationship holds for the cross-section, except that it is scaled with linear dimension as its square. Therefore,

$$S = S_0 \left(1 + \alpha_l \left(T - T_0\right)\right)^2 .$$

As for the resistance, we know that for constant dimensions of the wire, it should increase as

$$R = R_0 \left(1 + \alpha_R \left(T - T_0\right)\right) .$$

Since the only variable that can change if the wire dimensions are constant is the resistivity, and moreover, the resulting resistance is directly proportional to it, it must follow

$$\rho = \rho_0 \left(1 + \alpha_R \left(T - T_0\right)\right) .$$

Now all that remains is to substitute into the known relation for the resistance of the wire

$$\begin{aligned} R &= \rho \frac{l}{S} , \\ R &= \rho_0 \left(1 + \alpha_R \left(T - T_0\right)\right) \frac{l_0 \left(1 + \alpha_l \left(T - T_0\right)\right)}{S_0 \left(1 + \alpha_l \left(T - T_0\right)\right)^2} , \\ R &= R_0 \frac{1 + \alpha_R \left(T - T_0\right)}{1 + \alpha_l \left(T - T_0\right)} \approx 1.35\,\Omega . \end{aligned}$$

In reality, however, the coefficients of thermal expansion are orders of magnitude negligible compared to the temperature coefficients of electrical resistivity. Therefore, in practice, it is not necessary to account for changes in the dimensions of the wire if we only want to determine its resistance.

Šimon Pajger
legolas@fykos.org

Problem 19 ... golf on the hill 5 points

A golfer was confident about hitting the ball across the whole course. However, his hands were shaking during the broadcast, and his drive was not successful at all. He underhanded the ball so that it came out at an angle $\alpha = 80^\circ$ to the horizontal plane at speed $v = 8.0\,\mathrm{m\cdot s^{-1}}$. The launch was made in the direction down the hill, which has a deviation from the horizontal direction of $\beta = 10^\circ$. The disappointed golfer turned back and did not look to see how far from the point of drive the ball landed. Therefore, calculate this figure.

That day, Kuba lost another ball...

Let's introduce a Cartesian coordinate system with the center at the point of the ball drive. In the direction of the x axis, no force acts on the ball, but in the direction of the y axis, the gravitational force gives it a negative acceleration g. Hence, at time t the position of the ball is $x = v_0 t \cos\alpha$ and $y = v_0 t \sin\alpha - \frac{1}{2}gt^2$. From the equation for x, we express the time t and substitute it into the equation for y. By this, we get rid of the parameter t and obtain the equation of the parabola along which the ball is moving

$$t = \frac{x}{v_0 \cos\alpha} \Rightarrow y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2(\alpha)}\,.$$

The plane of the hill has the direction $\tan(180° - \beta) = -\tan\beta$, so it is described by the equation $y = -\,x\tan\beta$. By comparing the equations of the parabola and the hill plane, we get their intersections, i.e. the point of drive $x = 0$ and the point of impact

$$x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2(\alpha)} = -x\tan\beta\,.$$

We want to get the horizontal distance of the point of impact where $x \neq 0$ (the case $x = 0$ would correspond to the point of drive), so we divide the equation by x

$$\frac{gx}{2v_0^2\cos^2(\alpha)} = \tan\alpha + \tan\beta$$

$$x = \frac{2v_0^2\cos^2(\alpha)\,(\tan\alpha + \tan\beta)}{g}\,.$$

Now all that is left is to express the distance d we're looking for using the horizontal distance x. Notice that the following holds

$$\cos\beta = \frac{x}{d}\,.$$

The resulting distance corresponds to

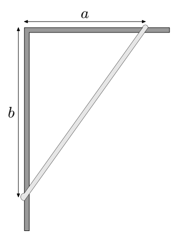
$$d = \frac{x}{\cos\beta} = \frac{2v_0^2\cos^2(\alpha)\,(\tan\alpha + \tan\beta)}{g\cos\beta} \doteq 2.3\,\mathrm{m}\,.$$

Jakub Smolík
jakub.smolik@fykos.org

Problem 20 ... rubberband around the corner — 4 points

We put a massless rubber band around the corner of the L-shaped rigid body. What is the smallest coefficient of friction between the rubber band and the solid that will prevent it from contracting it? The rubber band bends over the edges of the body at points $a = 5.0\,\mathrm{cm}$ and $b = 7.0\,\mathrm{cm}$ from the corner.

They're gonna put Jarda on a rack.



We can think of a rubber band as of a set of two springs (each on one side of the body), both of which start and end at the points where the rubber band touches the object. Let us denote these points as A and B. The two springs together exert a force F at point A towards point B (and vice versa).

To prevent the rubber band from contracting, the frictional force at both points must be greater than the elastic force. The frictional force at point A is $F_{\mathrm{TA}} = fF\cos\alpha$, while at point B, it is $F_{\mathrm{TB}} = fF\sin\alpha$. For angle α holds $\tan\alpha = a/b$.

The force trying to pull the rubber band down equals $F_\mathrm{A} = F\sin\alpha$ at point A, and $F_\mathrm{B} = F\cos\alpha$ at point B. For the rubber band to stay still, the following must hold

$$fF\cos\alpha \geq F\sin\alpha\,,$$
$$fF\sin\alpha \geq F\cos\alpha\,.$$

If we multiply the first inequality by f and substitute it into the second one, we get the condition $f \geq 1$.

In our case, $\alpha < 45^\circ$. Then the first inequality is satisfied for all $f \geq 1$. However, the second inequality implies

$$f \geq \frac{\cos\alpha}{\sin\alpha} = \frac{b}{a} = 1.4\,.$$

So the coefficient of friction must be at least 1.4.

Jaroslav Herman
jardah@fykos.org

Problem 21 ... level sensor

6 points

Consider a vessel with a non-conductive liquid of relative permittivity $\varepsilon_\mathrm{r} = 1.80$, in which two vertical partially immersed parallel plates form a plate capacitor. The capacitor is connected to a coil so that together they are part of the LC oscillator circuit. How many times does the natural frequency of this oscillator increase when the plates of the capacitor are quarterly submerged, compared to the case when the vessel is empty? *I bet that, just like Vašek, you have already thought about a way of measuring the level in a non-mechanical way.*

The frequency of the LC oscillator can be determined by the relationship

$$f = \frac{1}{2\pi\sqrt{LC}}\,,$$

where the capacitance of a plate capacitor with a plate area S spaced by a distance d in the air is given by

$$C = \varepsilon_0\frac{S}{d}\,.$$

When we immerse the plates into the liquid, we can view the device as two capacitors in parallel – one with a plate area of $3S/4$, the other with a plate area of $S/4$. Their capacitance will then be given by the sum of their individual capacitances due to the parallel connection. It will thus apply

$$C' = \varepsilon_0\frac{3S}{4d} + \varepsilon_0\varepsilon_\mathrm{r}\frac{S}{4d} = \varepsilon_0\frac{S}{d}\frac{3+\varepsilon_\mathrm{r}}{4}\,.$$

Now we can determine the ratio of the natural frequencies as

$$\frac{f'}{f} = \sqrt{\frac{C}{C'}} = \sqrt{\frac{4}{3+\varepsilon_\mathrm{r}}} \doteq 0.913\,.$$

So the natural frequency is reduced. Let's mention that it's actually *a capacitive liquid level sensor.*

Vojtěch David
vojtech.david@fykos.org

Problem 22 ... mountain biking reloaded 4 points

Matěj is riding on a mountain bike trail, passing oncoming cyclists. At the top of the hill, exhausted cyclists are moving at an average speed of $5\,\mathrm{km\cdot h^{-1}}$ *in both directions, and Matěj meets there* 0.02 *oncoming bikes per second on average. In contrast, at the lowest point of the bike trail, cyclists are revved up (from both directions) and ride at an average speed of* $50\,\mathrm{km\cdot h^{-1}}$. *How many oncoming bikes do Matěj encounter here on average? The bike trail has no turnoffs, and Matěj always moves at a speed of* $10\,\mathrm{km\cdot h^{-1}}$.

Matěj mountain biked while reloading.

Let us denote the frequency f_0 with which cyclists pass a static observer in one direction. If cyclists move at an average speed of v at a given location, then their average linear density on the trail is $\lambda = f_0/v$. Let's denote Matěj's speed by u. Relative to the static observer, Matěj passes $u\lambda$ more cyclists per second. Therefore, Matěj's frequency of passing oncoming cyclists is $f = f_0 + u\lambda = f_0\,(1 + u/v)$.

We have two equations (for the top and bottom of the hill) from which we can exclude the unknown f_0

$$f_{\mathrm{top}} = f_0\left(1 + \frac{u}{v_{\mathrm{top}}}\right)\,,$$

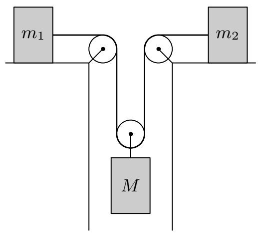
$$f_{\mathrm{bot}} = f_0\left(1 + \frac{u}{v_{\mathrm{bot}}}\right)\,,$$

which gives us the desired frequency

$$f_{\mathrm{bot}} = f_{\mathrm{top}}\frac{v_{\mathrm{top}}(v_{\mathrm{bot}} + v)}{v_{\mathrm{bot}}(v_{\mathrm{top}} + v)} = 0.008\,.$$

Matěj Mezera
m.mezera@fykos.org

Problem 23 ... a pit with pulleys 5 points



We have a pit in which a block with mass $M = 84.6\,\mathrm{kg}$ *is hanging from a movable pulley. On the horizontal surface to the left of the pit, there is a block with mass* $m_1 = 26.4\,\mathrm{kg}$, *and to the right of the pit, there is another block with mass* $m_2 = 33.8\,\mathrm{kg}$. *To each of these is tied one end of a rope, with a movable pulley hanging in the pit. There are, of course, two pulleys on the edges of the pit so that the rope runs only horizontally and vertically.*

Assume that everything moves without friction; the rope and pulleys are immaterial. What is the magnitude of acceleration the block in the pit will move with?

Lego felt like he does not create enough problems with pulleys.

Since both the rope and the pulleys are immaterial, the total force acting on each element of the rope must be zero (because $F = ma$). Therefore, the rope is stretched along its entire length by the same force; let us denote it by T. Since the gravity of blocks m_1 and m_2 is canceled out by the normal force of the ground, the tensile force from the rope is also the resultant force acting on them. Thus, block m_1 will accelerate with $a_1 = T/m_1$ towards the pit and similarly, the acceleration of m_2 will be $a_2 = T/m_2$.

The block inside the pit is pulled downwards by its gravity of magnitude $F_g = Mg$, and upwards by the tensile force of the rope T on both sides, so in total $2T$. Its resultant acceleration downwards is therefore $a = g - 2T/M$.

How are these accelerations related? If we moved the first block by x_1 and the second block by x_2, both towards the pit, then the block in the pit (because it is on a movable pulley between those two blocks) will move by the average of these two displacements (for intuition, we can easily check this on cases when $x_1 = x_2$ or $x_2 = 0$). Symbolically $x = (x_1 + x_2)/2$. When we derive this equation twice by time, we get the equation for acceleration $a = (a_1 + a_2)/2$ which we are looking for.

We plug all calculated accelerations into this equation and express T

$$\begin{aligned} g - 2\frac{T}{M} &= \frac{\frac{T}{m_1} + \frac{T}{m_2}}{2} \\ g &= T\left(\frac{2}{M} + \frac{1}{2m_2} + \frac{1}{2m_1}\right) \\ T &= \frac{g}{\frac{2}{M} + \frac{1}{2m_2} + \frac{1}{2m_1}}\,. \end{aligned}$$

What remains is to plug T back into the formula for the acceleration of the block inside the pit, and we have the result

$$a = g - 2\frac{T}{M} = g - 2\frac{g}{\frac{2M}{M} + \frac{M}{2m_2} + \frac{M}{2m_1}} = g\left(1 - \frac{1}{1 + \frac{M}{4m_2} + \frac{M}{4m_1}}\right) = 5.77\,\mathrm{m{\cdot}s^{-2}}\,.$$

Šimon Pajger
`legolas@fykos.org`

Problem 24 ... effective trapezoid 5 points

Consider an electrical source that provides a trapezoidal voltage such that for the first third of the period, the voltage increases linearly from $0.00\,\mathrm{V}$ *to* $5.00\,\mathrm{V}$*, then remains constant at* $5.00\,\mathrm{V}$ *for a third of the period, and then decreases linearly back to* $0.00\,\mathrm{V}$ *in the last third of the period. We will connect the source to a resistor. What constant-voltage source could we replace the original source with to have the same average power on the component (i.e., determine the effective voltage of the source)?*

Karel was varying the problem on effective values.

The mean power of the AC source is determined by integrating the instantaneous power over one period (and dividing by this period), i.e.

$$\frac{U_{\mathrm{ef}}^2}{R} = \overline{P} = \frac{1}{T}\int_0^T P\,\mathrm{d}t\,.$$

In our case, we can write instantaneous power as

$$P = \frac{U^2}{R}\,,$$

where the voltage U has the following waveform on the time interval $[0, T]$

$$U = \begin{cases} \frac{3t}{T} \cdot 5\,\mathrm{V} & t \in [0, T/3)\,\mathrm{s} \\ 5\,\mathrm{V} & t \in [T/3, 2T/3]\,\mathrm{s} \\ \frac{3(T-t)}{T} \cdot 5\,\mathrm{V} & t \in (2T/3, T]\,\mathrm{s} \end{cases}$$

Thus, we can substitute and break it down into three integrals. Then we get

$$U_{\mathrm{ef}} = \sqrt{\frac{1}{T}\left(\int_0^{T/3} \frac{225t^2}{T^2}\,\mathrm{d}t + \int_{T/3}^{2T/3} 25\,\mathrm{d}t + \int_{2T/3}^{T} \frac{225(T-t)^2}{T^2}\,\mathrm{d}t\right)}\,\mathrm{V} =$$

$$= \sqrt{\frac{450}{T^3}\int_0^{T/3} t^2\,\mathrm{d}t + \frac{25}{3}}\,\mathrm{V} =$$

$$= \frac{5\sqrt{5}}{3}\,\mathrm{V} \doteq 3.73\,\mathrm{V}\,.$$

Vojtěch David
vojtech.david@fykos.org

Problem 25 ... race against time 4 points

You are sitting in a car that is traveling at a constant velocity of $80\,\mathrm{km{\cdot}h^{-1}}$*, and you enter a* $5\,\mathrm{km}$ *long tunnel with no cellular signal. At the same time, you are watching a stage of the Tour de France on your cell phone with* $3.5\,\mathrm{km}$ *to go. The cyclists are going at* $40\,\mathrm{km{\cdot}h^{-1}}$ *and just as your signal drops out, they begin to descend, accelerating steadily until the finish line. What is their largest possible acceleration so that you can see the last* $30\,\mathrm{s}$ *of the stage when you exit the tunnel and get the signal again?* *Dávid was watching cycling in the car.*

When setting up the equation for the motion of cyclists, we first need to realize how long it will take the cyclists to arrive at the finish line under the given conditions. It is clear that this time (t_c) is equal to the sum of the time it takes the car to pass through the tunnel (t_a) and the reserve time (t_r), which expresses how much spare time the cyclists shall have after exiting the tunnel to reach the finish line. The mathematical expression looks as follows

$$t_{\mathrm{c}} = t_{\mathrm{a}} + t_{\mathrm{r}} = \frac{s_{\mathrm{a}}}{v_{\mathrm{a}}} + t_{\mathrm{r}} = 255\,\mathrm{s}\,. \tag{3}$$

Now we can set up an equation for the distance traveled by the cyclists, and then use this equation to express the acceleration $a_{\max}$

$$s_{\mathrm{c}} = v_{\mathrm{c}}t_{\mathrm{c}} + \frac{a_{\max}t_{\mathrm{c}}^2}{2} \quad \Rightarrow \quad a_{\max} = \frac{2\left(s_{\mathrm{c}} - v_{\mathrm{c}}t_{\mathrm{c}}\right)}{t_{\mathrm{c}}^2}\,. \tag{4}$$

For the general expression of the result, we insert the expression we got in (3) into equation (4)

$$a_{\max} = \frac{2\left(s_{\mathrm{c}} - v_{\mathrm{c}}\left(\frac{s_{\mathrm{a}}}{v_{\mathrm{a}}} + t_{\mathrm{r}}\right)\right)}{\left(\frac{s_{\mathrm{a}}}{v_{\mathrm{a}}} + t_{\mathrm{r}}\right)^2} = 0.021\,\mathrm{m{\cdot}s^{-2}}\,.$$

The result is therefore $a_{\max} = 0.021\,\mathrm{m{\cdot}s^{-2}}$.

Dávid Brodňanský
david.brodnansky@fykos.org

Problem 26 ... escape across the frozen lake

4 points

A migrant tries to flee the country in winter, and his path leads across a frozen lake. The border guards are on his tail, so he must plan his route as efficiently as possible. There is a direct national border that runs across the lake, beyond which he will be safe. There are thick reeds all around the shore, but suddenly he sees a free area of ice. He runs onto the frozen surface at a speed of $u = 4.3\,\mathrm{m\cdot s^{-1}}$ *in a direction parallel to the border from which he is* $d = 38\,\mathrm{m}$ *away. How long will it take him, at the minimum, to reach safety if the coefficient of friction between the ice and his boots is* $f = 0.05$ *and no other obstacles stand in his way?*

Jarda thinks these are the real problems in Eastern Europe right now.

The maximal force he can act in any horizontal direction is $F = fmg$. From Newton's second law, we get an equation for his maximal acceleration as $a = gf$.

His net acceleration must point perpendicular to the border while he wants to cross the border as fast as possible. So he moves uniformly accelerated, and his distance in that direction is $s = \frac{1}{2}at^2$. After substituting $a = gf$ and $s = d$ we get

$$t = \sqrt{\frac{2d}{gf}} = 12.4\,\mathrm{s}\,.$$

Jaroslav Herman
jardah@fykos.org

Problem 27 ... heating the helium

5 points

An unknown volume V_0 *of helium gas is stored at atmospheric pressure and at a temperature* $T_0 = 297\,\mathrm{K}$*. How large is this volume if, by supplying heat* $Q = 42\,\mathrm{J}$ *at constant pressure, the temperature of the gas is increased by* $\Delta T = 2.5\,\mathrm{K}$*?*

Karel was browsing the textbooks and hadn't seen this one.

In an isobaric process (i.e. process at constant pressure), the equation of state

$$pV = nRT$$

shows that temperature and volume will be directly proportional. If the temperature is increased from T_0 to $T_1 = T_0 + \Delta T$, then the volume is increased from V_0 to

$$V_1 = V_0 \frac{T_1}{T_0} = V_0 \left(1 + \frac{\Delta T}{T_0}\right)\,.$$

We can thus immediately calculate the work done in the process. Since $\mathrm{d}W = p\,\mathrm{d}V$ and the pressure does not change in this process, we only need to multiply the pressure by the change in volume and the work done is

$$W = p_\mathrm{a}\Delta V = p_\mathrm{a} V_0 \frac{\Delta T}{T_0}\,.$$

However, we have to realize that the supplied heat Q is converted into both an increase in the internal energy of the gas and the carried-out work. We know that the internal energy of a gas is proportional to nRT (so it is also proportional to pV). But what is the constant of proportionality? The equipartition theorem says that this constant is $n/2$, where n is the number of degrees of freedom for the motion of the molecules. For a monatomic gas, $n = 3$ as the molecule can move independently in the x, y, z directions. Since helium is a monatomic gas, the internal energy of the gas has increased by

$$\Delta U = U_1 - U_0 = \frac{3}{2}nRT_1 - \frac{3}{2}nRT_0 = \frac{3}{2}p_\mathrm{a}\Delta V\,.$$

By substituting into the first law of thermodynamics, we get

$$Q = W + \Delta U = p_\mathrm{a}V_0\frac{\Delta T}{T_0} + \frac{3}{2}p_\mathrm{a}V_0\frac{\Delta T}{T_0}\,,$$

$$V_0 = \frac{2QT_0}{5p_\mathrm{a}\Delta T} = 1.97\cdot 10^{-2}\,\mathrm{m^3}\,.$$

Šimon Pajger
legolas@fykos.org

Problem 28 ... tropical climate — 5 points

Viktor wants his house plants to thrive, so he decided to increase the humidity in the room. Before turning on the humidifier, the humidity in the room was 30 %. The humidifier can evaporate 5 ml of water every minute. At what percentage would the humidity in the room stabilize if Viktor opened the window and every minute, 3 % of air was replaced in the room? Assume that the outside atmosphere has the same humidity as the air inside the room before the device was turned on. Viktor's room has dimensions 4 × 5 × 2.5 m, and the air temperature is 25 °C. *Viktor wanted to experience what it was like to live in the rainforest.*

Let us denote the initial relative humidity in the room Φ_0 and the relative humidity at steady state Φ_e. If the relative humidity of the room is steady state, the total mass of water in the air in the room does not change. Looking at the balance, a humidifier adds water vapor to the air every minute with a mass of

$$m_1 = V_0\rho\,,$$

where $V_0 = 5\,\mathrm{ml}$ is the volume of water evaporated by the humidifier in one minute and $\rho = 997\,\mathrm{g{\cdot}cm^{-3}}$ is the density of water at 25 °C. At the same time, however, ventilation exchanges $\eta = 3\,\%$ of the more humid air from the room for less humid air from outside. To find out what the mass m_2 of water vapor that leaves the room per minute by this process is, we need to know two equations. The mass of water vapor m in the room air and the absolute humidity θ are linked by the equation

$$\theta = \frac{m}{V}\,,$$

where V is the volume of the room. The relative humidity is then calculated from the absolute humidity as

$$\Phi = \frac{\theta}{\theta_\mathrm{n}}\,,$$

where θ_n is the absolute humidity of the air saturated with water vapor (in that case the relative humidity is 100 %). This value is temperature dependent, at the given 25 °C is $\theta_\mathrm{n} = 23\,\mathrm{g{\cdot}m^{-3}}$. Using these two equations for absolute and relative humidity, we can write

$$m_2 = \eta abc(\theta_e - \theta_0)\,,$$

$$m_2 = \eta abc\theta_\mathrm{n}(\Phi_e - \Phi_0)\,,$$

where abc is the volume of the room. In equilibrium, $m_1 = m_2$. Thus we get

$$V_0\rho = \eta abc\theta_\mathrm{n}(\Phi_e - \Phi_0)\,.$$

By rearranging, we express the steady-state relative humidity Φ_e as

$$\Phi_e = \Phi_0 + \frac{V_0\rho}{\eta abc\theta_\mathrm{n}}\,.$$

Substituting the numeric values we get $\Phi_e \doteq 44\,\%$.

Daniela Dupkalová
daniela@fykos.org

Problem 29 ... The Pit and the Pendulum

6 points

In a cylindrical pit that is 30 cm *wide, a small ball is suspended on a* 50 cm *long massless string. The point of suspension lies on the symmetry axis of the pit. We give the ball a horizontal velocity v such that it bounces perfectly elastically off the vertical walls twice before returning to its original position. It will not pass through any point in space twice during this process and will have the same velocity vector at the end as at the beginning. Determine the magnitude of the velocity v.* *Jarda happened to be paying attention in literature class.*

Once the horizontal velocity is given, the ball starts to move in a circle as far as the string to which it is attached allows. After a perfectly elastic bounce from the wall, the vertical component of the velocity is conserved. The horizontal component will change the sign but not the direction. Hence, the ball will move back towards the axis of symmetry of the cylindrical pit. Then a second bounce occurs, and the ball moves back again. Since we require that it does not pass through any point twice during this process, it will move through the air along the parabola after the first bounce. After the second bounce, it will again begin to follow the arc of the circle.

The law of reflection applies for bounces, so the angles of the trajectories (circle and parabola) are the same with respect to the vertical axis.

We calculate the angle of incidence α from the geometry of the problem as

$$\sin\alpha = \frac{d}{2l}\,,$$

where d is the width of the pit and l is the length of the pendulum.

Let us denote the velocity of the ball after reflection as v_o. Then, it will move along a parabola characterized by the following equation

$$y = x\tan\alpha - \frac{g}{2}\frac{x^2}{v_\mathrm{o}^2\cos^2\alpha}\,,$$

where y is the vertical coordinate, x is the horizontal coordinate in the direction from the wall, and g is the acceleration due to gravity. The bounce occurs for $y = 0$ and $x = d$, so from the equation we express v_o as

$$v_\mathrm{o} = \sqrt{\frac{gd}{2\sin\alpha\cos\alpha}}\,.$$

Yet, we still have to determine the velocity of the ball at the lowest point from the law of conservation of energy

$$v = \sqrt{v_\mathrm{o}^2 + 2gl\left(1 - \cos\alpha\right)} = \sqrt{\frac{gd}{2\sin\alpha\cos\alpha} + 2gl\left(1 - \cos\alpha\right)}$$
$$= \sqrt{\frac{gl}{\sqrt{1 - \left(\frac{d}{2l}\right)^2}} + 2g\left(l - \sqrt{l^2 - \left(\frac{d}{2}\right)^2}\right)} = 2.4\,\mathrm{m{\cdot}s^{-1}}\,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem 30 ... magneto-resistor 5 points

Let us assemble an electric circuit. We have a long straight wire of resistance $R = 57\,\mathrm{m\Omega}$. In parallel to the wire, we connect a magneto-resistor, i.e., a component that changes its resistance depending on the external magnetic field. The distance between the magneto-resistor and the wire is $d = 3.5\,\mathrm{cm}$, and its resistance at zero magnetic field is $r_0 = 85\,\Omega$. In parallel to the wire, we connect a power supply with voltage $U = 12\,\mathrm{V}$. A current $I = 140\,\mathrm{mA}$ flows through the magneto-resistor after it has stabilized. Assume that its resistance varies linearly with the magnetic field B over a given range. Determine the coefficient of proportionality.

Jarda got lost in a physicists' warehouse.

The resistance of a magneto-resistor changes as $r = r_0 + \alpha B$, where

$$\alpha = \frac{r - r_0}{B}$$

is the parameter we want to determine. The dependency of the magnetic field B on the electric current flowing through the wire can be represented by the following formula

$$B = \frac{\mu_0 I_R}{2\pi d} = \frac{\mu_0 U}{2\pi d R}\,,$$

where μ_0 is the permeability (of vacuum), and I_R is the current flowing through a resistor.

Resistance r_0 is known from the problem statement, and resistance r is equal to U/I. After substituting the values to the previous equation, we get

$$\alpha = \frac{2\pi}{\mu_0} dR\left(\frac{1}{I} - \frac{r_0}{U}\right) \doteq 590\,\Omega{\cdot}\mathrm{T}^{-1}\,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem 31 ... wheel against the direction of travel

5 points

What fraction of the total travel time does the velocity (component parallel to the ground) of one fixed point of the outermost edge of the railway wheel (the flange) have the opposite direction as the train itself? Consider that the wheel overhangs the track by 35 mm *and its diameter with the flange is* 1 400 mm. *Karel was thinking about trains.*

The time when the flange is moving against the direction of the travel depends only on the diameter of the wheel $d = 1\,400\,\mathrm{mm}$ and the overhang of the flange $l = 35\,\mathrm{mm}$. A point on the outermost edge of the wheel makes a uniform rectilinear motion in the direction of the travel, and a uniform motion along the circle centered in the axis of the wheel. The angle between this point and the horizontal axis is angle α. In order to express both motions using the given values, we introduce ω, which is an angular velocity at which the wheel rotates, and it is constant. Using that, we express the velocity of the rectilinear motion in the direction of travel v_1

$$v_1 = \omega\left(\frac{d}{2} - l\right) .$$

The velocity of the point on the edge of the wheel as it moves along the circle v_2 is

$$v_2 = \omega\frac{d}{2} .$$

The velocity vector v_2 is always perpendicular to the position vector of the point on the edge of the wheel. We decompose it into two vectors, one of which is parallel to the velocity vector v_1, which is the one we are interested in when it has an opposite direction to v_1. We denote it as v_3. If the magnitude of this vector is greater than the magnitude of v_1, it means, that the point on the edge of the wheel is moving against the direction of travel, while its horizontal motion is the sum of vectors v_1 and v_2. Using the goniometric functions we express v_3 from v_2.

$$v_3 = v_2\cos\left(\frac{\pi}{2} - \alpha\right) = \omega\frac{d}{2}\cos\left(\frac{\pi}{2} - \alpha\right) = \omega\frac{d}{2}\sin\alpha .$$

Now we set v_1 to be equal to v_3 and see for what angles α the equality holds.

$$\omega\frac{d}{2}\sin\alpha = \omega\left(\frac{d}{2} - l\right)$$

$$\sin\alpha = \frac{\frac{d}{2} - l}{\frac{d}{2}} .$$

We get $\alpha_1 = 1.253\,\mathrm{rad}$ and $\alpha_2 = 1.888\,\mathrm{rad}$. In the interval between those two angles, the velocity v_3 is greater than the velocity v_2, and the point on the edge of the wheel is moving against the direction of travel. It remains to find out how much of the total time this is.

$$\frac{\alpha_2 - \alpha_1}{2\pi} = 0.10 .$$

From the result, we see that the point at the edge of the wheel moves in the opposite direction for a tenth of the total time.

Josef Knápek
`josef.knapek@fykos.org`

Problem 32 ... too high frequency

6 points

Let's have a resistor with resistance $R = 11\,\mathrm{k\Omega}$ and a capacitor with capacitance $C = 2\,\mathrm{\mu F}$ connected in series to the AC voltage source. For what value of frequency in the circuit will the voltage amplitude across the capacitor drop to one tenth of the maximum achievable value?

Jarda doesn't like quick changes.

The impedance of a resistor is simply equal to R, while for a capacitor it is defined as $\frac{-i}{C\omega}$, where i is the imaginary unit and $\omega = 2\pi f$ is the angular frequency of the source. We will solve the problem using complex numbers.

The total impedance in series is $Z = R - \frac{i}{C\omega}$. The complex current in the circuit will thus be $\frac{U}{Z}$. The voltage drop on the resistor os RI and the capacitor is left with

$$U_C = U - RI = U\left(1 - \frac{R}{Z}\right) = U\frac{-\frac{i}{C\omega}}{R - \frac{i}{C\omega}} = U\frac{-i}{RC\omega - i}\,.$$

We are interested in the amplitude of the voltage across the capacitor, which is determined by the absolute value of U_C according to the relation

$$|U_C| = \frac{U}{\sqrt{(RC\omega)^2 + 1}}\,.$$

The maximum voltage across the capacitor clearly occurs at zero frequency (DC), which is $U_{\mathrm{max}} = U$. From the condition in the problem statement, we get the equation

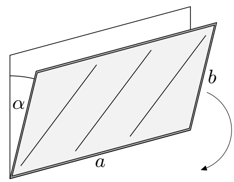
$$\frac{|U_C|}{|U|} = \frac{1}{10} \quad \Rightarrow \quad f = \frac{\sqrt{99}}{2\pi RC} \doteq 72\,\mathrm{Hz}\,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem 33 ... attention, the window is falling

7 points

The rectangular window is open on its upper side so that its plane makes an angle of 10° with the vertical direction. However, due to strong wind, the latch holding it in place is released, and the window begins to rotate about its lower horizontal axis before hitting the wall below. At what angular velocity does it hit the wall? The glass alone has an area density $\rho = 15\,\mathrm{kg{\cdot}m^{-2}}$, dimensions $a = 130\,\mathrm{cm}$ by $b = 60\,\mathrm{cm}$, and is embedded in a frame of mass $m_{\mathrm{r}} = 4\,\mathrm{kg}$ (do not consider its dimensions). Consider the whole window to be planar and homogenous in parts.



It's unfortunate. Jarda has to find a glazier, but at least he has a problem for Physics Brawl Online.

We solve the problem using the law of conservation of energy. The potential energy converts to rotational about the bottom axis according to the formula

$$Mg\Delta h = \frac{1}{2}J\omega^2\,,$$

where M is the total mass of the window, g is gravity, Δh is the difference in positions of the center of gravity at the beginning and the end of the motion, J is the moment of inertia with respect to the rotational axis, and ω is the angular velocity of the window at impact.

The difference in positions is given by

$$\Delta h = \frac{(1+\cos\alpha)}{2} b\,,$$

where $b = 60\,\mathrm{cm}$ is the height of the window and α is the initial angle of opened window. The two in the denominator is there because the center of gravity of the window is in the center of its height.

The total mass of the window is $M = m_\mathrm{r} + m_\mathrm{s} = m_\mathrm{r} + ab\rho$.

The total moment of inertia is given by the sum of the moment of inertia of the glass and the frame. If we look at the window from the side, we see it as a rotating line. The moment of inertia of a rod rotating about one of its ends is $ml^2/3$, where m is the mass of the rod, and l is its length. In our case, it does not matter if a rod or a rectangle is rotating (they look the same from the side). Therefore, the moment of inertia of the glass is $J_\mathrm{s} = ab\rho b^2/3$.

For the frame, the situation is a bit more difficult. It is divided into four segments, one of which is not rotating, the next two look like rods when looking from the side, and the last one (originally at the top) looks like a point. When calculating the moments of inertia of individual segments, we introduce the linear density of the frame as

$$\lambda = m_\mathrm{r}/(2a+2b)\,.$$

The moment of inertia of the original upper part of the frame is then equal to $m_\mathrm{rh} b^2 = \lambda a b^2$, and for the side parts of the frame together

$$2\cdot\frac{1}{3} m_\mathrm{rb} b^2 = \frac{2}{3}\lambda b^3\,.$$

The total moment of inertia of the frame is therefore

$$J_\mathrm{r} = \lambda\left(ab^2 + \frac{2}{3}b^3\right) = m_\mathrm{r} b^2 \frac{a+\frac{2}{3}b}{2a+2b}\,.$$

Now we have all we need to write the final result as

$$\omega = \sqrt{\frac{2Mg\Delta h}{J}} = \sqrt{\frac{3\left(m_\mathrm{r}+ab\rho\right)g\left(1+\cos\alpha\right)}{m_\mathrm{r} b\frac{3a+2b}{2a+2b} + a\ \rho b^2}} = 9.5\,\mathrm{rad{\cdot}s^{-1}}\,.$$

Jaroslav Herman
jardah@fykos.org

Problem 34 ... blocks on top of each other on a plane 5 points

Two blocks of masses 600 g *and* 700 g *are placed on top of each other on an inclined plane which is deviated from the horizontal direction by* 30°. *The heavier of them is on the bottom, and the interface between them is perfectly smooth. The coefficient of static friction between the lower block and the plane is* 0.3, *while the dynamic friction coefficient is* 0.2. *How large is the angle* *between the force exerted by the plane on the block and the inclined plane itself?*

Marek found the block to be lonely on an inclined plane.

Let us denote the mass of the upper block m_1 and the bottom one m_2. We denote the angle of the inclined plane as α. The force exerted by the inclined plane on the block can be decomposed into the direction perpendicular and parallel to the plane. Let us denote the perpendicular force as F_n and the parallel as F_t. The F_t is realized by friction and is, therefore, proportional to F_n.

If the force corresponding to the static friction "holds" the bottom block, $F_\mathrm{t} = 0.3F_\mathrm{n}$ holds; if not, $F_\mathrm{t} = 0.2F_\mathrm{n}$ will hold.

Since the upper block cannot fall through the lower block, and the lower block cannot fall through the plane, the following equation holds

$$F_\mathrm{n} = (m_1 + m_2)g\cos\alpha\,.$$

We denote the force acting on the lower block against the force F_t as F_1. The following then holds

$$F_1 = m_2 g\sin\alpha\,.$$

The condition that the static friction "holds" the lower block corresponds to the inequality

$$\begin{aligned} 0.3F_\mathrm{n} &> F_1\,,\\ 0.3 &> \frac{m_2}{m_1+m_2}\tan\alpha\,,\\ 0.3 &> 0.31\ldots\,, \end{aligned}$$

which does not hold so the bottom block moves.

We denote the angle we are looking for as β. For it

$$\tan\beta = \frac{F_\mathrm{N}}{F_\mathrm{t}} = \frac{F_\mathrm{N}}{0.2F_\mathrm{N}} = \frac{1}{0.2}$$

holds, and thus finally

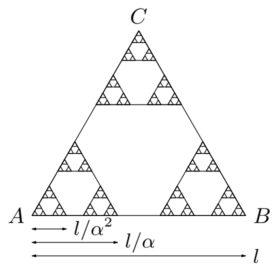
$$\beta = \arctan\frac{1}{0.2} \doteq 78.7^\circ\,.$$

Marek Milička
marek.milicka@fykos.org

Problem 35 ... long power line

8 points

Consider a wire with constant electrical resistance $r_0 = 1\,\Omega$ per unit length l. Let us construct a resistive network from this wire as shown in the diagram. We start from a large equilateral triangle with side length l. Let us connect the points on the sides of this triangle at a distance l/α from the nearest vertex with a wire (where $\alpha \geq 2$), making an α times smaller triangle at each vertex of the large triangle. We repeat the same procedure for these smaller triangles and so on indefinitely (each step produces triangles α times smaller than in the previous step). Finally, we connect the source terminals to vertices A and B. What will the resistance of this network be if $\alpha = 4$?



Radka couldn't find the end of the extension cord.

Using the triangle-star transformation, any triangle with smaller triangles at its vertices can be transformed into an equivalent simple triangular circuit.

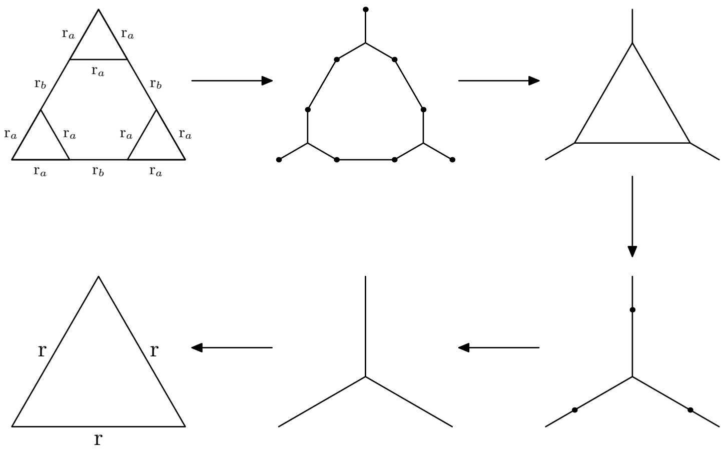


Fig. 1: Transforming circuits.

Specifically, if we use the notation of resistances in the figure (r_a is the resistance of one side of a small triangle and r_b is the resistance of one side of the large triangle minus twice the side of a small triangle), we obtain the formula

$$r = \frac{5}{3} r_a + r_b \,.$$

If we tried constructing the network by gradually adding smaller triangles at the vertices of larger triangles, the equivalent resistance between two vertices of the largest triangle would follow the formula

$$r_{i+1} = \frac{5}{3\alpha} r_i + \left(1 - \frac{2}{\alpha}\right) r_0 \,.$$

where the coefficient $1/\alpha$ in the first term accounts for the way new triangles get smaller than previously added ones. In the limit of infinitely many steps, we may use $r_i = r_{i+1} = r_\mathrm{t}$, where r_t is the resistance of one side of an equilateral triangle which would be equivalent to the whole network. From the equation above, we get

$$r_\mathrm{t} = \frac{3(1 - \frac{2}{\alpha})}{3 - \frac{5}{\alpha}} r_0$$

and for $\alpha = 4$

$$r_\mathrm{t} = \frac{6}{7}\,\Omega \,.$$

The resulting resistance between vertices A and B is

$$R = \frac{2}{3} r_\mathrm{t} = \frac{2(1 - \frac{2}{\alpha})}{3 - \frac{5}{\alpha}} r_0 = \frac{4}{7}\,\Omega \,.$$

Alternative solution

We won't use the triangle-star transformation or limits, but superposition of networks. Let the current flowing through the bottom wire with resistance $r_1 = r_0 \left(1 - \frac{2}{\alpha}\right)$ be I_1 and the current flowing through the two diagonal wires with resistance r_1 be I_2. Imagine the whole network as a superposition of a network without the bottom wire and another network without the diagonal wires - voltage in each vertex is the sum of voltages in the two circuits. We express the voltage between vertices A and B using equations

$$V = 2\left(\frac{I_1}{2} + I_2\right)\frac{R}{\alpha} + 2I_2 r_1 + I_2 \frac{R}{\alpha}\,,$$
$$V = 2\left(I_1 + \frac{I_2}{2}\right)\frac{R}{\alpha} + I_1 r_1\,,$$
$$V = (I_1 + I_2)\,R\,.$$

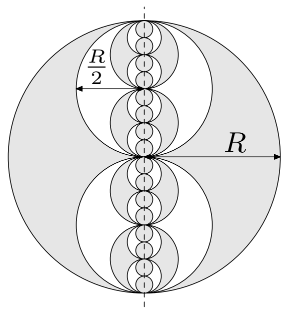
Subtracting the first and second equation, we get $(2I_2 - I_1)\frac{r_t}{\alpha} = (I_1 - 2I_2) r_1$. For positive resistances, we must then have $I_1 = 2I_2 = \frac{2V}{3R}$; plugging this back in and dividing by positive V gives the same result

$$R = \frac{2}{3 - \frac{5}{\alpha}} r_1 = \frac{2(1 - \frac{2}{\alpha})}{3 - \frac{5}{\alpha}} r_0\,.$$

Radka Křížová
radka.krizova@fykos.org

Problem 36 ... a ball for Jáchym — 6 points

We will create a special ball for Jáchym in successive steps. We begin with a homogenous solid ball of radius R and material density ρ, and choose one major axis passing through the center of the ball. We remove two smaller balls of radii $R/2$ from the large ball, whose centers lie on the major axis and are at a distance $R/2$ from the center of the large ball so that the resulting holes are adjacent to each other. Then we add 4 balls inside with radii $R/4$. Their centers lie on the major axis, and we place the balls next to each other. We continue in the same way. That is, we make two spherical holes inside each of the small balls, and then put half-radius balls in them again. We continue this way to infinity.



What is the ratio of the total mass of the ball for Jáchym and the mass of the full ball with the radius R and the same density ρ? *Karel wanted Jáchym to get his special ball, too.*

Note that in each step, we add/remove twice as many balls as in the previous step. These balls all have their radius halved. Their total volume (and thus, due to the constant density, also their total mass) will therefore be $2 \cdot (1/2)^3$ times the volume of the balls added/removed in the previous step. If we denote by M the mass of the ball we start with, we get the mass of the ball for Jáchym as a sum of an infinite series

$$m = M - \frac{M}{4} + \frac{M}{16} - \frac{M}{64} + \cdots = \sum_{i=0}^{\infty} M \left(\frac{-1}{4}\right)^i\,,$$

which we can add up using

$$\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}\,,$$

as

$$m = \frac{4}{5}M\,.$$

Since we want three significant figures in the result, the answer is 0.800.

Vojtěch David
vojtech.david@fykos.org

Problem 37 ... polarization as far as it goes — 6 points

Karel decided to put five linear polarizers behind each other. He chose the angle between the planes of the first and second polarizer to be α, the next $\alpha - \pi/2$, then α again, and so on. What is the maximum intensity of light that Karel can obtain in this way if he shines on the first polarizer with unpolarized light of intensity I_0? Express as the ratio $I_{\max}/I_0$.

Petr was thinking about stacking polarizing filters.

We assume that $0 \le \alpha \le \pi/2$ because we are only examining the angle between the planes. Also, note that even equality is out of the question while the polarizing filters perpendicular to each other will screen out 100 % of the radiation.

When incident on the first polarizer, the light is linearly polarized, so we observe a 1/2 decrease in intensity. Furthermore, if the polarized light falls on a polarizing filter rotated by an angle α, we can determine the intensity I' of the transmitted radiation as

$$I' = I\cos^2(\alpha)\ .$$

The first polarizer only polarizes the light, which we have already described above, so let us now examine the losses on the other four polarizers. There, we can use the identity $\cos(\alpha - \pi/2) = -\sin(\alpha)$ for the total measured intensity I to write

$$I = \frac{I_0}{2}\cos^2(\alpha)\cos^2\left(\alpha - \frac{\pi}{2}\right)\cos^2(\alpha)\cos^2\left(\alpha - \frac{\pi}{2}\right) = \frac{I_0}{2}\cos^4(\alpha)\sin^4(\alpha)\ .$$

It remains to find the maximum of this expression. By deriving and setting it equal to zero, we get

$$\cos^3(\alpha)\sin^3(\alpha)\left(\cos^2(\alpha) - \sin^2(\alpha)\right) = 0\,,$$

which for $0 < \alpha < \pi/2$ is equivalent to

$$\cos(2\alpha) = 0\,,$$

where $\alpha = \pi/4$. Now all we have to do is add and we get

$$\frac{I_{\max}}{I_0} = \frac{1}{32} = 0.03125\,.$$

Vojtěch David
vojtech.david@fykos.org

Problem 38 ... dog walking

7 points

Jarda is walking his dog. They are walking down a long straight street. Jarda's distance from the garden fences is 5.0 m *while his dog is tied to a* 4.0 m *long leash. Already from afar, the dog knows about his friend Žeryk, who is lying contently in a kennel by the fence of one of the gardens and is looking forward to a barking encounter. For this reason, he always tries to be as close to Žeryk as possible on his leash. What is the highest velocity Jarda's dog will move at? Jarda walks with a constant velocity* $v = 1.0\,\mathrm{m{\cdot}s^{-1}}$.

Jarda invents problem assignments even while walking his dogs.

Let us introduce a Cartesian coordinate system with its center at the point where Jarda stands when he is closest to Žeryk, i.e. perpendicular to the fence at a distance of 5 m. Let $t = 0$ correspond to the moment when Jarda passes through this point. Jarda's dog is thus exactly 1 m from Žeryk.

At the time t, Jarda's position is $x = vt$. Let us determine the position of Jarda's dog. The dog is always on the line between Jarda and the point $x = 0, y = D = 5\,\mathrm{m}$ where Žeryk is. For the angle between this line and the x axis, the following applies

$$\sin\varphi = \frac{D}{\sqrt{D^2 + v^2t^2}}\,,\; \cos\varphi = \frac{vt}{\sqrt{D^2 + v^2t^2}}\,.$$

Then the position of Jarda's dog is

$$x_\mathrm{p} = vt - r\frac{vt}{\sqrt{D^2 + v^2t^2}}\,,\; y_\mathrm{p} = r\frac{D}{\sqrt{D^2 + v^2t^2}}\,.$$

By deriving with respect to time, we find the components of its velocity

$$\dot{x}_\mathrm{p} = v\left(1 - \frac{r}{\sqrt{D^2 + v^2t^2}} + \frac{v^2t^2r}{\sqrt{D^2 + v^2t^2}^3}\right)\,,\; \dot{y}_\mathrm{p} = -\frac{rDv^2t}{\sqrt{D^2 + v^2t^2}^3}\,.$$

After multiplying and summing the two components, we obtain

$$v_\mathrm{p}^2 = \frac{v^2}{\left(D^2 + v^2t^2\right)^2}\left(\left(D^2 + v^2t^2\right)^2 + r^2D^2 - 2rD^2\sqrt{D^2 + v^2t^2}\right)\,.$$

We derive the expression and set it equal to zero. The solution to this equation is either $t = 0$ or it leads to a simple condition

$$3\sqrt{D^2 + v^2t^2} = 2r\,,$$

which cannot be satisfied for any real t. At the time $t = 0$ the dog's velocity is minimal, and no other extreme value is attained. Thus the maximal velocity of the dog will be $1.0\,\mathrm{m{\cdot}s^{-1}}$ at times $t = \pm\infty$.

Jaroslav Herman
jardah@fykos.org

Problem 39 ... ordinary estimate of mean air height 6 points

Consider a hollow cylinder with base area $S = 327\,\mathrm{cm}^2$, which is closed at the bottom and infinitely tall at the top. The whole cylinder is placed in a homogeneous field with a gravitational acceleration g. Let us fill it with diatomic nitrogen molecules with temperature $T_0 = 25\,{}^\circ\mathrm{C}$ so that the pressure $p_0 = 101\,\mathrm{kPa}$ is at the bottom. What is the mean height of these molecules above the base of the cylinder? Assume that nitrogen behaves like an ideal gas.

Lego neglected as much as possible.

Classical solution

Of course, we will use the ideal gas equation of state $pV = NkT$. From this we can immediately see that in a small air layer of height $\mathrm{d}h$, located at height h, where the pressure is $p(h)$; there will be

$$\mathrm{d}N(h) = \frac{p(h)\,S}{kT_0}\,\mathrm{d}h$$

molecules. When we find out exactly how the pressure (and hence the "linear density" of the particles) evolves with height, we will be able to find their mean height as

$$\langle h\rangle = \frac{\int_0^\infty h\,\mathrm{d}N(h)}{\int_0^\infty \mathrm{d}N(h)} = \frac{\int_0^\infty hp(h)\,\mathrm{d}h}{\int_0^\infty p(h)\,\mathrm{d}h}\,.$$

We have a given pressure at the bottom, so we need to find some (obviously differential) relationship for the decreasing pressure with the upward direction. If we take a layer of molecules and divide its total gravity by the area S, we get the pressure it exerts on the gas below. The pressure below this layer will be greater than the pressure above it by this amount. Let's move from words to equations. Let's denote the mass of one molecule m. Then the described pressure difference will be

$$\mathrm{d}p = -\frac{mg}{S}\,\mathrm{d}N(h) = -\frac{mg}{kT_0}p(h)\,\mathrm{d}h\,.$$

This is a differential equation that can be solved simply by the separation of variables. However, you need to watch out for the correct sign – the pressure will decrease with increasing height! So we solve the equation

$$\frac{1}{p}\,\mathrm{d}p = -\frac{mg}{kT_0}\,\mathrm{d}h\,,$$

$$\int_{p_0}^{p(h)}\frac{1}{p}\,\mathrm{d}p = \int_0^h -\frac{mg}{kT_0}\,\mathrm{d}h\,,$$

$$\ln\left(\frac{p(h)}{p_0}\right) = -\frac{mgh}{kT_0}\,,$$

$$p(h) = p_0\exp\left(-\frac{mgh}{kT_0}\right)\,.$$

We get that the pressure will decrease exponentially with height. We plug the obtained expression into the formula for the mean height that we provided above

$$\langle h\rangle = \frac{\int_0^\infty hp_0\exp\left(-\frac{mgh}{kT_0}\right)\mathrm{d}h}{\int_0^\infty p_0\exp\left(-\frac{mgh}{kT_0}\right)\mathrm{d}h} = \frac{\left(\frac{kT_0}{mg}\right)^2}{\frac{kT_0}{mg}} = \frac{kT_0}{mg} = 9020\,\mathrm{m}\,.$$

The mean height of the molecules above the bottom base of the cylinder is, therefore 9015 m. We may notice that it does not depend on p_0 at all. This makes sense since one of the fundamental properties of an ideal gas is that the molecules do not affect each other. Hence, no matter how many molecules we add, the mean height won't change... Now someone might ask if we couldn't obtain the result in a simpler way. The answer is that we could.

Statistical solution

Probably the best-known relationship from statistical physics is the Boltzmann distribution. This tells us that the probability that some degree of freedom (for example, the height of a single molecule h) will have a particular value is proportional to $\exp(-E/kT)$, where E is the energy associated with that value. In this case, it is the potential energy of the molecule, so $E(h) = mgh$. Then the average height of one molecule is obtained immediately as

$$\langle h \rangle = \frac{\int_0^\infty h \exp\left(-\frac{mgh}{kT_0}\right) \mathrm{d}h}{\int_0^\infty \exp\left(-\frac{mgh}{kT_0}\right) \mathrm{d}h} \,.$$

We see that we get the same integrals as at the end of the classical solution. The denominator (also called sum over states Z) is there again due to normalization.

Finally, we would add that the result in the order of higher thousands of meters makes quite a lot of sense if we realize that Mount Everest is usually climbed with oxygen equipment, but it is possible to climb it even without it. So the characteristic air height could be in the order of 8000 m.

Šimon Pajger
`legolas@fykos.org`

Problem 40 ... modern art 7 points

We have 2 rectangular cuboids, one with mass $M = 9.50\,\mathrm{kg}$ and the other with mass m. We lift the second one and press it against the wall by the first one, which is standing on the ground. What is the largest possible mass m to prevent the second cuboid from falling if the coefficient of friction between the cuboids, the cuboid and the wall, and between the cuboid and the ground is $f = 0,288$? Assume that the rectangular cuboid standing on the ground is long enough not to topple backward. *Lego really didn't know how to name this problem.*

A rectangular cuboid of mass m is pressed upwards only by friction forces – between the cuboid and the wall and between the cuboids themselves. The friction force can be calculated by multiplying the friction coefficient f (which is the same for all pairs of surfaces in this problem) with the normal force that presses the two surfaces together. Since the cuboids are not moving in the horizontal direction, the forces in this direction must be balanced. Thus, the normal force on one side of the lifted cuboid must be the same as on the other. If we denote this force by F_1, the friction force on each side will be $F_{t1} = fF_1$. The sum of the friction forces must compensate for the weight of the cuboid, so $mg = 2F_{t1}$. From here, we can express the force by which the cuboids push on each other as $F_1 = mg/(2f)$.

It comes from Newton's 3rd law of motion that cuboid M is pressed away by force F_1. To remain standing, a force of the same magnitude must act on it in the opposite direction. The only other force acting on it in the horizontal direction is the friction force from the ground. Its magnitude must therefore be equal to F_1, and again, we can express it as the product of f

and the normal force. However, here comes the trickiest part of the problem – the normal force will not only be Mg, but we have to consider Newton's third law. The friction force between the cuboids acts both on the cuboid held in the air and on the cuboid of mass M that holds it there (and the same is true for the wall).

So $F_2 = Mg + F_{t1} = Mg + mg/2$, and then the friction force between the ground and the cuboid is $F_{t2} = fg(M + m/2)$. We set this friction force equal to F_1 and get the equation for the maximum mass m, at which the lifted cuboid will not fall yet

$$F_1 = F_{t2}$$
$$\frac{mg}{2f} = fg\left(M + \frac{m}{2}\right)$$
$$\frac{m}{f} - fm = 2fM\,,$$

from that

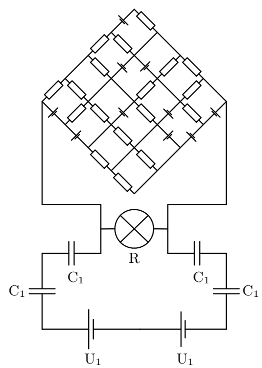
$$m = 2M\frac{1}{\frac{1}{f^2} - 1} = 1.72\,\mathrm{kg}\,.$$

Šimon Pajger
`legolas@fykos.org`

Problem 41 ... maze of resistance 6 points

All resistors (including the bulbs) in the circuit in the figure have a value of $R = 1.0\,\Omega$, capacitors have a capacitance of $C = 1.0\,\mathrm{F}$ and all batteries have a DC voltage of $U = 1.0\,\mathrm{V}$. Calculate the current through the bulb when the current is settled.

Marek J. encountered resistance while walking.



The circuit can be separated into upper and lower parts. The lower part consists of a voltage source and capacitors only. We know that DC current will not flow through the capacitors. Therefore, after initially charging the capacitors and rearranging the charge to compensate for the voltage, the situation settles down (zero current), and we can ignore the lower part of the circuit in our analysis of the problem.

The upper part, however, is really a maze of resistors and batteries. After wandering for a while, however, we can see that there is a path of least resistance for electric current (literally)! This is the part of the circuit where there are only batteries (and "empty" parts with just the wire). The batteries have differently oriented polarities, but since the value of their voltage is always the same and constant, the total voltage on that part of the circuit is ultimately the simple sum of the positive and negative voltages.

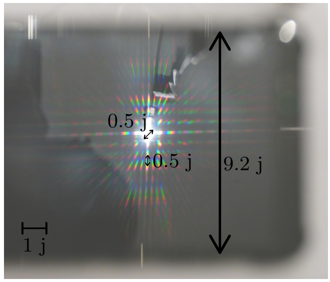
Looking at the figure, we get the total voltage value $U_c = 3\,\mathrm{V}$. The loop with the least resistance in the circuit is unambiguous and is completed by the bulb in parallel. Thus, from Kirchhoff's second law, we know the voltage across the bulb, which is equal to the aforementioned three volts. Finally, using Ohm's law, we calculate the current through the bulb as

$$I = \frac{U_c}{R} = 3.0\,\mathrm{A}\,.$$

Marek Jankola
marekj@fykos.org

Problem 42 ... screen resolution

8 points



Jarda noticed a diffraction on his cell phone, so he tried to calculate the size of the individual pixels, which he thought were squares. He placed the cell phone 1.05 m *under a light source with a size of* 1.5 cm *and photographed the reflection on the screen with a camera whose lens was* 40 cm *above it. In the photograph, the width of the screen was* 9.2 j*, the diameter of the light was* 0.5 j*, and the distance of the yellow maxima was* 0.5 j*, where* 1 j *is the scale on the photograph. The screen of Jarda's phone has dimensions* 13 cm *and* 7.5 cm*. How many pixels are there?*

Jarda introduced an experimental problem to Physics Brawl Online.

First, let's note that the cell phone screen will act as a diffraction grating but also as a mirror that will show the source "behind the screen". From the grating equation

$$d \sin \alpha = k\lambda$$

we calculate the grating constant (i.e., the pixel size) d. Since we are working with small angles, we use the approximation $\alpha \approx \sin \alpha \approx \tan \alpha$ and measure the angle $\Delta\alpha$ between two adjacent maxima. Then we get

$$d = \frac{\lambda}{\Delta\alpha}\,,$$

where $\lambda = 580\,\mathrm{nm}$ is the wavelength of the yellow color.

The only task left is to determine $\Delta\alpha$ from the geometry of the apparatus and the photograph. To do this, we use the given lengths of the important objects in the photograph and the known width of the cell phone. The phone is a mirror, so we can imagine that the light source is behind it and the screen acts only as a diffraction grating. Consider that the light comes from parallel rays from behind the screen, where it is deflected due to diffraction and interference. Therefore, we are only interested in what happens in front of the display.

Consider the camera lens as the optical system that projects the incoming rays onto the sensor. For simplicity, we replace the camera lens in the diagram with a lens with focal length f. The sensor is in such a plane behind the focal point that the image is in focus. It is a known fact that the position of the image behind the optical system (i.e., the position in the photograph) is proportional to the angle at which the rays enter the system (at least in the geometric optics approximation). Since we don't need to have the center of the phone exactly on the optical axis, we'll just use angle differences and distance differences in the image. From the knowledge of the real width of the cell phone and its size in the picture, we find a proportionality constant, which we use to calculate $\Delta\alpha$.

The difference in the angles taken by the rays coming from the two edges of the phone is

$$\beta = \frac{s}{h} = \frac{7.5\,\mathrm{cm}}{40\,\mathrm{cm}}\,,$$

where s is the width of the screen and h is the distance of the lens from the screen. This angle is proportional to the width of the display in the photo $s_{\mathrm{f}} = 9.2\,\mathrm{j}$.

Now consider the rays from two adjacent yellow maxima. The difference in their angles (relative to the optical axis, for example) is exactly $\Delta\alpha$. This angle difference is, in turn, proportional to the distance in the photo $s_z = 0.5\,\mathrm{j}$. Therefore, we can relate the angles β and $\Delta\alpha$ to the positions in the photograph and we get

$$\frac{s_f}{\beta} = \frac{s_z}{\Delta\alpha} .$$

We have already expressed β from the true dimensions of the objects. So now we are all set to calculate d as

$$d = \lambda \frac{s_f}{s_z \beta} = \lambda \frac{h s_f}{s_z s} = 57\,\mathrm{\mu m} ,$$

where $\lambda = 580\,\mathrm{nm}$ is the wavelength of yellow light.

The number of these units on the whole display is thus given by the ratio of the area of the whole display, which is sv, where $v = 13\,\mathrm{cm}$, and the area of one pixel d^2. The numerical value is

$$N = \frac{sv}{d^2} = 3.0 \cdot 10^6 .$$

We adjusted the display dimensions in the problem statement from 12.8 cm and 6.4 cm compared to the experiment. With these values, we would get a result of $1.9 \cdot 10^6$. The stated resolution of the mobile is 2160 by 1080, so there are $2.3 \cdot 10^6$ pixels. The difference between these values is due to the error in the measurement of the height h or an approximation of the ray that passes through the lens.

Jaroslav Herman
jardah@fykos.org

Problem 43 ... wired   7 points

Two adjacent electric transmission towers are distanced $s = 400\,\mathrm{m}$ in a flat landscape. One of the wires between them is fixed on both ends at a height of $H = 20\,\mathrm{m}$ above the ground. In winter, when the air temperature is $t_z = -10\,\mathrm{^\circ C}$, the lowest point of the wire is due to sagging at $h_z = 19\,\mathrm{m}$ above ground. At what height will this point be in summer when the temperature is $t_l = 20\,\mathrm{^\circ C}$? The coefficient of thermal expansion of the steel from which the wire is made is $\alpha = 13 \cdot 10^{-6}\,\mathrm{K^{-1}}$. For simplicity, neglect the elongation of the wire due to its weight.

Matej Rz was outside.

It is a well-known fact that the shape of the sagging wire corresponds to the so-called catenary, i.e. the curve described by the equation

$$r(x) = a \cosh\left(\frac{x}{a}\right) ,$$

where a is a parameter specifying the "sag of the rope", the lowest point of the catenary is thus defined as the point $[0, a]$. We will need to be able to determine the length of the catenary – we can find this either by solving the simple integral,[2] or we can state it as a known statement. Since the problem is symmetric, the length of our catenary will be

$$d = a \sinh\left(\frac{b}{2a}\right) - a \sinh\left(\frac{-b}{2a}\right) = 2a \sinh\left(\frac{b}{2a}\right) ,$$

[2] $\int_\alpha^\beta \sqrt{1 + (a \cosh'(x/a))^2}\,\mathrm{d}x$

where b denotes the distance between the ends of the catenary. Next, we introduce a natural coordinate system with the x axis parallel to the ground and the y axis parallel to the transmission towers, placing its origin at a distance a "under" the top of the chain. This gives us a natural way to work with the catenary using the approach described above.

We determine the length of the catenary in winter. It holds $b = s$ and $r(b/2) = a + H - h_z$. From this, we can already determine the parameter a and then the length d. Unfortunately, we have to solve the equation for a numerically

$$a + H - h_z = a \cosh\left(\frac{s}{2a}\right) \quad \Rightarrow \quad a \doteq 20\,000.166\,7\,\mathrm{m}\,,$$

from where then

$$d = 2a \sinh\left(\frac{s}{2a}\right) \doteq 400.007\,\mathrm{m}\,.$$

The length of d' in summer is easily determined as

$$d' = (1 + \alpha(t_l - t_z))\, d = (1 + \alpha(t_l - t_z))\, 2a \sinh\left(\frac{s}{2a}\right)$$

To determine the lowest point of the wire, we must now apply the same procedure as above backward. First, we introduce a coordinate system with its origin in $[0, a']$, determine the parameter a' numerically from the new length of the string, and finally calculate the height h_l as the difference $H - (r'(b/2) - a')$, where $r'(x)$ describes the shape of the catenary in summer. It applies

$$2a' \sinh\left(\frac{s}{2a'}\right) = (1 + \alpha(t_l - t_z))\, 2a \sinh\left(\frac{s}{2a}\right)\,,$$

which we have to solve numerically again, but we get quite straightforwardly

$$a' \doteq 4\,049.097\,\mathrm{m}\,.$$

Finally, we can write

$$h_l = H - a' \cosh\left(\frac{s}{2a'}\right) + a' \doteq 15.1\,\mathrm{m}\,.$$

In our solution, we rounded the intermediate results, while for the final solution we substituted values with accuracy to multiple significant digits.

Vojtěch David
vojtech.david@fykos.org

Problem 44 ... electromagnet

8 points

Consider a long thin bar made from a material with high relative permeability $\mu_r = 7\ 000$. A wire is wrapped around the rod to obtain a solenoid with $N = 1\ 000$ turns. The solenoid is then bent into a circle so that the opposite ends of the bar almost touch, but a narrow gap of width $s = 1.00\,\mathrm{mm}$ is left between them. This gap is small compared to both the radius of the circle $r = 20.0\,\mathrm{cm}$ and the radius of the bar. If a current $I = 500\,\mathrm{mA}$ flows through the wire, determine the magnitude of the magnetic induction in the gap.

Jindra stole the problem from a textbook.

From Maxwell's equations it is possible to prove the continuity of the perpendicular component of the magnetic induction $\mathbf{B}$ and the parallel component of the magnetic field strength $\mathbf{H}$ at the interface of two materials (see the proof below).

In the gap, the magnetic induction is perpendicular to the surface of the bar, so the magnetic induction B there will be the same as everywhere else in the solenoid. The magnetic intensity in the core of the coil is H_{in} and the magnetic intensity in the gap is H_m. We know that $\mu_r \mu_0 H_\mathrm{in} = = \mu_0 H_\mathrm{m} = B$.

The radius of the bar is negligible compared to the radius of the circle r. The line integral along the core of the coil along a circle of radius r gives

$$NI = \oint \mathbf{H} \cdot \mathrm{d}\mathbf{l} \,,$$

$$NI = H_\mathrm{in}(2\pi r - s) + H_\mathrm{m} s \,,$$

$$NI = \frac{B}{\mu_r \mu_0}(2\pi r - s) + \frac{B}{\mu_0} s \,,$$

$$B = \frac{\mu_r \mu_0 N I}{2\pi r + (\mu_r - 1)s} = 0.533\,\mathrm{T} \approx 0.53\,\mathrm{T} \,.$$

We can notice that the magnetic field in the gap with this approximation is determined by the width of the gap and is greatly amplified compared to the situation where we would use a coil without a core (for a coil without a core we use the same relation, we substitute $\mu_r = 1$ and $s = 0\,\mathrm{mm}$, and get $B' = 5.00 \cdot 10^{-4}\,\mathrm{T} \ll B = 0.533\,\mathrm{T}$).

Proof of the continuity of the perpendicular component $\mathbf{B}$ *and the tangential component* $\mathbf{H}$ *at the interface*

For the proof, we need two of Maxwell's equations for the magnetic field

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0, \qquad \oint \mathbf{H} \cdot \mathrm{d}\mathbf{l} = I + \int \frac{\partial \mathbf{D}}{\partial t} \cdot \mathrm{d}\mathbf{S} \,.$$

The first equation states that there are no magnetic monopoles. The second equation links the line integral of the magnetic field strength to the current flowing through that closed curve and the time variation of the electric induction flux $\mathbf{D}$.

Consider two different materials in contact with relative magnetic permeabilities μ_1 and μ_2. Let us take a closer look at a small section of their contact area so that we can neglect the curvature of the interface and treat it as a plane surface.

Around the planar interface, we form a Gaussian surface – a cube with one base lying in material 1 and the other base lying in material 2. The bases with surface A are parallel to the material interface. The cube has a negligible height h (see the figure on the left). From the equation for the absence of magnetic monopoles, we derive that the perpendicular component of the magnetic induction vector is continuous at the interface of any two materials

$$0 = \oint \mathbf{B} \cdot \mathrm{d}\mathbf{S} \approx A\mathbf{B}_1 \cdot \mathbf{n}_1 + A\mathbf{B}_2 \cdot \mathbf{n}_2 = -AB_{1,\perp} + AB_{2,\perp} \,,$$

$$B_{1,\perp} = B_{2,\perp} \,.$$

When deriving, we neglected the flux of magnetic induction through the side walls of the cube, as we are working in the limit $h \to 0$. This way we proved a fact on which the solution to the problem relies. For interest, we shall also derive the continuity of the tangential component of the magnetic intensity vector.

To prove that the tangential component of the magnetic intensity vector is continuous at the interface, we draw a closed curve around the interface – a rectangle of length a and negligible height h. The sides of length a are parallel to the interface (see the figure on the right). For the proof, we use the equation

$$\oint \mathbf{H}\cdot \mathrm{d}\mathbf{l} = I + \int \frac{\partial \mathbf{D}}{\partial t}\cdot \mathrm{d}\mathbf{S}$$

If we push the height h to zero, the current I flowing through the rectangle and the magnetic flux $\int (\partial \mathbf{D}/\partial t)\cdot \mathrm{d}\mathbf{S}$ will also go to zero. This follows from the fact that they are both proportional to the area of the rectangle, which goes to zero in the limit. However, the integral $\oint \mathbf{H}\cdot \mathrm{d}\mathbf{l}$ on the left-hand side remains, as it depends on the perimeter of the rectangle, which remains non-zero

$$\oint \mathbf{H}\cdot \mathrm{d}\mathbf{l} = 0 \approx \mathbf{H}_1\cdot \mathbf{a}_1 + \mathbf{H}_2\cdot \mathbf{a}_2 = aH_{1,\parallel} - aH_{2,\parallel}\,,$$

$$H_{1,\parallel} = H_{2,\parallel}\,.$$

The contribution from the vertical sides of the rectangle has been neglected since we have sent their height h to zero.

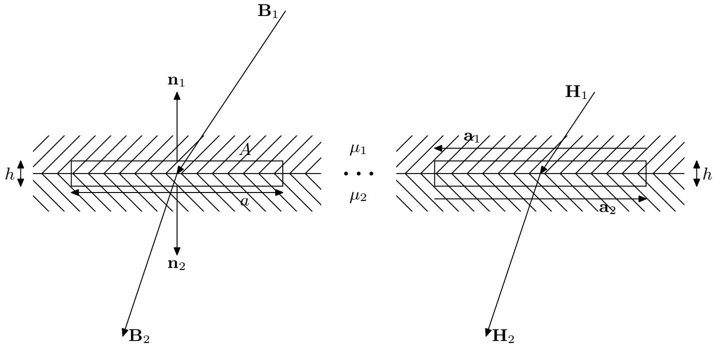


Fig. 2: For the proof of the continuity of the tangential and normal components.

Jindřich Jelínek
jjelinek@fykos.org

Problem 45 ... ladder 7 points

A ladder leaning against a wall has a length of $l = 5\,\mathrm{m}$ and a mass of $m = 10\,\mathrm{kg}$, with its center of gravity located exactly at its midpoint. When Jindra stands on the ladder, his center of gravity is at a distance of $h = 4\,\mathrm{m}$ along the ladder from the ground, and a distance of $x = 30\,\mathrm{cm}$ perpendicularly to it. Jindra's mass is $M = 70\,\mathrm{kg}$. The coefficient of kinetic friction between the ladder and the floor is $f_1 = 0.4$ and between the ladder and the wall is $f_2 = 0.6$. Determine the maximum angle between the ladder and the vertical wall, before Jindra starts falling. *Jindra painted the walls of a room.*

The amount of data complicates the problem. Let's analyze all the forces at work. At the contact with the ground, a normal force N_1 (size unknown) acts perpendicularly upwards on the ladder and a frictional force T_1 (size unknown) acts towards the wall. At the point of contact with the wall, a normal force N_2 (magnitude unknown) and a frictional force T_2 (magnitude unknown) act on the ladder perpendicular to the wall. The downward gravitational force Mg, where g is the acceleration due to gravity, acts on Jindra, and the downward gravitational force mg acts on the ladder.

If the linked system of Jindra and the ladder is supposed to be in equilibrium, the sum of all forces and the sum of all torques must be zero

$$(M+m)g = N_1 + T_2\,,$$

$$T_1 = N_2\,,$$

$$Mg(h\sin\alpha - x\cos\alpha) + mg\frac{l}{2}\sin\alpha = lT_2\sin\alpha + lN_2\cos\alpha\,,$$

$$Mg(l\sin\alpha - h\sin\alpha + x\cos\alpha) + mg\frac{l}{2}\sin\alpha + lT_1\cos\alpha = lN_1\sin\alpha.$$

The first equation represents the balance of forces in the vertical direction, the second equation describes the balance of forces in the horizontal direction, the third equation represents the balance of moments of forces with respect to the point of contact of the ladder with the ground, and the fourth equation describes the balance of moments of forces with respect to the point of contact of the ladder with the wall. There are five unknowns in the system of four equations, the forces N_1, T_1, N_2, T_2, and the angle α. However, we are only interested in the specific maximal value of the angle $\alpha = \alpha_{max}$ at which the ladder starts to slip.

Let's think about what is happening at that point. The ladder starts sliding on the ground and along the wall at a critical moment. This means that none of the frictional forces T_1, T_2 can prevent the ladder from moving. The maximum magnitudes of the friction forces are $T_1 = f_1N_1$ and $T_2 = f_2N_2$. Thus, these two equations hold at the largest possible angle α_{max}. Our four equations reduce to two (taking advantage of the fact that $T_2 = f_2N_2 = f_1f_2N_1$)

$$(M+m)g = (1+f_1f_2)N_1\,,$$

$$Mg(h\sin\alpha_{\max} - x\cos\alpha_{\max}) + mg\frac{l}{2}\sin\alpha_{\max} = lf_1f_2N_1\sin\alpha_{\max} + lf_1N_1\cos\alpha_{\max}\,.$$

Express N_1 from the first equation and plug it into the second equation

$$Mg(h\sin\alpha_{\max} - x\cos\alpha_{\max}) + mg\frac{l}{2}\sin\alpha_{\max} = \frac{f_1l(M+m)g}{1+f_1f_2}\left(f_2\sin\alpha_{\max} + \cos\alpha_{\max}\right)\,,$$

cancel g and convert the sine and cosine terms to each other

$$\left(Mx + \frac{f_1l(M+m)}{1+f_1f_2}\right)\cos\alpha_{\max} = \left(Mh + m\frac{l}{2} - \frac{f_1f_2l(M+m)}{1+f_1f_2}\right)\sin\alpha_{\max}\,.$$

We divide the equation by $\cos\alpha_{\max}$ (we're not afraid of dividing by zero) and we get

$$\tan\alpha_{\max} = \frac{Mx + \frac{f_1l(M+m)}{1+f_1f_2}}{Mh + m\frac{l}{2} - \frac{f_1f_2l(M+m)}{1+f_1f_2}} \doteq 33.4^\circ\,.$$

Jindřich Jelínek
jjelinek@fykos.org

Problem 46 ... mushroom hunting

7 points

Jarda went to an oak forest to collect parasol mushrooms. The wind was blowing quite strongly, so acorns from trees were falling with a frequency per unit area of $N = 1.0 \cdot 10^{-4}\,\mathrm{m^{-2}{\cdot}s^{-1}}$*. The area that Jarda normally occupies is* $A = 1100\,\mathrm{cm^2}$*. What is the probability that an acorn will hit him if he plans on picking mushrooms in the forest for* $t = 35\,\mathrm{min}$*?*

Jarda was hit by an acorn three times in a row!

From the problem statement, the mean value of the number of acorns that will fall on Jarda in 35 minutes can be directly determined as

$$\lambda = NAt\,.$$

Afterward, it is important to realize that the fall of an acorn is an independent event and that the number of independent events per unit of time is represented by the Poisson distribution. This distribution states that for a random variable X with a mean value of the number of occurrences of λ, the probability of x occurrences is

$$P(X = x) = \frac{\lambda^x}{x!}\mathrm{e}^{-\lambda}\,.$$

The probability that at least one acorn falls on Jarda is then given by

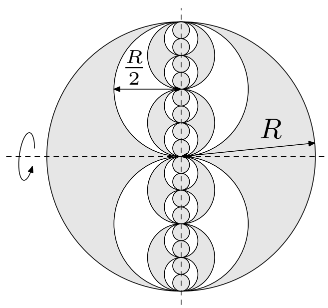
$$P(X \geq 1) = 1 - P(X = 0) = 1 - \mathrm{e}^{-\lambda} \doteq 0.023\,.$$

One could argue that this problem has little to do with physics, but the mathematical model – studying independent phenomena – is an inseparable part of nuclear or statistical physics.

Vojtěch David
vojtech.david@fykos.org

Problem 47 ... a ball for Jáchym reloaded

7 points



In successive steps, we will make a special ball for Jáchym. We start with a homogenous solid ball of radius R *and material density* ρ*. We choose one major axis passing through the center of the ball. Next, we extract two balls from the large ball that have radii* $R/2$*, whose centers lie on the major axis and are at distances* $R/2$ *from the center, such that the resulting holes are closely adjacent to each other. Then we add 4 balls of radii* $R/4$ *inside. Their centers again lie on the main axis and the balls are placed next to each other. We repeat this process of making two spherical holes in each of these smaller balls and then putting the half balls into these again ad infinitum. What is the moment of inertia of Jáchym's ball with respect to the axis perpendicular to the main axis? Express it as a ratio to the moment of inertia of a full ball of radius* R *and equal density* ρ*.*

Karel (even Vojta and the other Vojta) thinks that Jáchym deserves the ball not once, but twice.

The moment of inertia of a ball of mass m and radius r rotating about the axis through its centre corresponds to

$$J = \frac{2}{5}mr^2\,.$$

In order to solve the problem, in addition to knowing that the moments of inertia of bodies rotating with respect to the same axis can be added to obtain the total moment of inertia,[3] we will also need Steiner's theorem. The latter states that if we move the axis of rotation of a rigid body passing through a center of gravity of mass m to a distance r, the moment of inertia changes from J to

$$J' = J + mr^2\,.$$

To avoid having to laboriously add up the geometric series, we note that we can use the recurrence of the whole the situation. Specifically, Jáchym's ball is composed of one large ball of radius R, in which two spherical cavities are excavated containing four smaller Jáchym's balls (of quartered radii) whose centers are $R/4$ and $3R/4$ away from the center of the original ball.

Before we get to the actual calculation, we need to consider how the moment of inertia of a body scaled by a factor α (while maintaining a constant density) will change. The moment of inertia depends on the square of the distance from the axis of rotation and on the mass, which is proportional to the third power of the magnitude.[4] Therefore if we reduce all the dimensions of the body in proportion to the coefficient α, the moment of inertia drops by α^5. Therefore, Jáchym's ball of a quarter radius will have a rotational inertia of $1/4^5$ of the original rotational inertia.

Finally, we will also need the masses of the quarter-sized Jáchym's balls. From the problem statement of "a ball for Jáchym" we know that the mass of large Jáchym's ball is $4M/5$, so the mass of the reduced balls will be equal to $M/80$ (i.e. $1/4^3$ of the mass of the original ball, since the mass decreases with the third power of "size").

Now we can finally do the math. For the moment of inertia of Jáchym's ball J, following equation must hold

$$J = \frac{2}{5}MR^2 - 2\left(\frac{2}{5}\frac{M}{8}\left(\frac{R}{2}\right)^2 + \frac{M}{8}\left(\frac{R}{2}\right)^2\right) + 2\left(\frac{J}{4^5} + \frac{M}{80}\left(\frac{R}{4}\right)^2\right) + 2\left(\frac{J}{4^5} + \frac{M}{80}\left(\frac{3R}{4}\right)^2\right)\,,$$

since it must be equal to the moment of inertia of the full ball without the moments of inertia of the two smaller balls of halved radius, in turn increased by the moments of inertia of the four small Jáchym's balls – all modified by Steiner's theorem. Solving this equation will then give us

$$J = \frac{28}{85}MR^2 = \frac{14}{17}J_0\,,$$

where $J_0 = 2/5MR^2$ is the moment of inertia of the solid ball. Therefore, the answer to the question in the problem statement is $14/17 \doteq 0.82$.

We may incidentally realize that the same result is obtained if we use Steiner's theorem to "subtract" two Jáchym's balls of half the size from the full ball.

Vojtěch David
`vojtech.david@fykos.org`

Problem 48 ... up and down 8 points

Ping pong is played with balls of diameter $d = 4.0\,\mathrm{cm}$ and mass $m = 2.7\,\mathrm{g}$. After losing a match, Jarda took one of these balls and out of anger fired it vertically in the air. The ball reached

[3] This stems trivially from the fact that the moment of inertia is the integral of the square of the distance from the axis of rotation over the total mass of the body.

[4] More precisely, it is the integral of the square of the distance from the axis of rotation over the mass.

the height $H = 8.5\,\mathrm{m}$. *Determine the ratio of the time the ball was going up and the time it was falling down.* *Jarda tried to hit the window in his dorm with a ping-pong ball, but he failed.*

Right at the beginning of the problem, it is important to remember that the ping-pong ball is not a point mass, so the drag force of air must be considered. We will use the following formula for the whole problem

$$F_\mathrm{o} = \frac{1}{2} C S \rho v^2 \,,$$

where the designation of the individual quantities will be explained later. The drag force depends on the square of the velocity v because the motion of the ball is fast enough to cause turbulent flow.

It is obvious that the ball will be falling longer than it will be going up because it will travel the same distance, but its average speed will be lower due to the dissipation of energy due to the drag force.

We divide the trajectory into two parts - the flight upwards and the flight downwards; we have different equations for the motion of both. The process of solving the equations is left as an exercise for an interested reader; here, we give only the necessary results. The solution process is also possible to find on the Internet or by using one of the computing software.

The equation of motion for upward flight is

$$ma = -(mg + kv^2) \,,$$

where m is the mass of the ball, g is the gravitational acceleration, a is acceleration of the ball, v is its speed, and $k = CS\rho/2$ is a constant. Here $S = \pi d^2/4$, ρ is the density of air, and $C = 0.5$ for a ball. If we orient the velocity upwards, the acceleration upwards is negative.

The solution to this equation is

$$v = \sqrt{\frac{gm}{k}} \tan\left(\sqrt{\frac{gk}{m}}\,(T - t)\right) \,,$$

where T is the time when v is zero, so the ball is at the top of its path. By integration, we get

$$h = \frac{m}{k} \log\cos\left(\sqrt{\frac{gk}{m}}\,(T - t)\right) + K \,.$$

From the initial condition, where $h = 0$ in $t = 0$, we get the value of the integration constant as

$$K = -\frac{m}{k} \log\cos\left(T\sqrt{\frac{gk}{m}}\right) \,.$$

By substituting in time $t = T_1$, we get

$$H = -\frac{m}{k} \log\cos\left(T_1\sqrt{\frac{gk}{m}}\right) \,,$$

from where we express the time T_1 as

$$T_1 = \sqrt{\frac{m}{gk}} \arccos e^{\frac{-kH}{m}} \,.$$

For a downfall, the equation of motion has the form

$$ma = mg - kv^2\,,$$

where the acceleration points downwards. We get the velocity in the form

$$v = \sqrt{\frac{gm}{k}}\tanh\sqrt{\frac{gk}{m}}t\,,$$

where in time $t = 0$ the velocity is also zero. By integration, we get

$$h = H - \frac{m}{k}\log\cosh\sqrt{\frac{gk}{m}}t\,,$$

where we have determined the integration constant from the initial condition. We find the time T_2 for which the ball falls down as

$$\sqrt{\frac{m}{gk}}\operatorname{argcosh} e^{\frac{kH}{m}} = T_2\,.$$

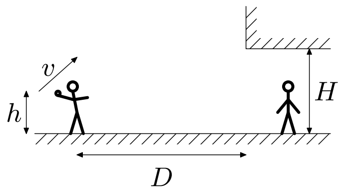
The solution to the problem is thus

$$\frac{T_1}{T_2} = \frac{\arccos e^{\frac{-kH}{m}}}{\operatorname{argcosh} e^{\frac{kH}{m}}} = 0.68\,.$$

Jaroslav Herman
jardah@fykos.org

Problem 49 ... revenge for the machines — 7 points

Jarda got really angry at Viktor for his noisy vacuum cleaners and started chasing him around the dormitory. Viktor is now hiding in a passage under the dorms. The ceiling is $H = 5.00\,\mathrm{m}$ high there. Jarda is standing at a distance $D = 12.0\,\mathrm{m}$ from the beginning of the passage when he throws an object at Viktor with velocity $v = 13.5\,\mathrm{m{\cdot}s^{-1}}$ from a height of $h = 2.10\,\mathrm{m}$. How far from the beginning of the passage must Viktor stand to avoid being hit by Jarda?



Viktor created an inspiring environment in his room...

The solution breaks down into several cases. If the velocity is small, it is not possible to hit the passage ceiling at any initial angle, so we do not have to reckon it. So we find the maximum distance to which an object can be hit from a height h at a given velocity.

For high velocities, it is worth throwing at a small angle so that the object does not rise too high and can get into the passage at all. For this case, it is important whether the vertex of the parabola along which the object is moving can be at a greater distance from Jarda than D. Let us explore this possibility. The equation of the parabola along which the object moves is

$$y = h + x\tan\alpha - \frac{g}{2}\frac{x^2}{v^2\cos^2\alpha}\,, \tag{5}$$

where g is the acceleration due to gravity, α is the initial angle of the trajectory with respect to the ground, y is the vertical coordinate upwards, and x is the horizontal coordinate towards the passage. At the apex of the parabola, y does not change, so the derivative of the previous function is zero, which allows us to find the position of the apex as

$$x_\mathrm{v} = \frac{v^2 \sin 2\alpha}{2g}\,,$$

where we used the goniometric identity $2 \sin\alpha \cos\alpha = \sin 2\alpha$. The maximum value of the sine function is 1, so the maximum possible value for x_v in our case is $9.3\,\mathrm{m}$, which is less than D. Thus, when entering the passage, the object will already be falling.

From equation 5 we find the distance of the point of impact from Jarda by setting $y = 0$ and solving the quadratic equation. We get

$$x_\mathrm{d} = \frac{v\cos\alpha}{g}\left(v\sin\alpha \pm \sqrt{v^2\sin^2\alpha + 2gh}\right)\,. \tag{6}$$

But at the same time, the condition $y(D) \leq H$ must be satisfied; otherwise, Jarda's object will end up hitting the roof of the passage. We find the extreme angles under which equality occurs; that is, the object's trajectory intersects the bottom corner of the passage roof. We plug $x = D$, $y = H$ into equation 5 and use the identity $1/\cos^2\alpha = 1 + \tan^2\alpha$. This gives us

$$0 = \frac{g}{2}\frac{D^2}{v^2}\tan^2\alpha_\mathrm{m} - D\tan\alpha_\mathrm{m} + \frac{g}{2}\frac{D^2}{v^2} + H - h$$

$$\Rightarrow \quad \tan\alpha_\mathrm{m} = \frac{v}{gD}\left(v \pm \sqrt{v^2 - 2g\left(\frac{gD^2}{2v^2} - h + H\right)}\right)\,.$$

The limiting angles are therefore $67.0\,^\circ$ and $36.6\,^\circ$. Between these angles, the object will not enter the passage.

It is possible that the maximum of function 6 does not lie in this interval of angles so that the object might not touch the mentioned corner of the passage at all. The angle for which the object will travel the furthest is found as

$$\alpha_\mathrm{max} = \arctan\frac{v}{\sqrt{v^2 + 2gh}} \doteq 42.1\,^\circ\,.$$

Since this angle lies in the forbidden interval, we would hit the passage. Since function 6 is first increasing and then decreasing for $\alpha > 0$, the longest possible range occurs for either $67.0\,^\circ$ or $36.6\,^\circ$. Substituting in 6, we find that the maximum range is for $36.6\,^\circ$, namely $20.3\,\mathrm{m}$. From this distance, we still need to subtract the distance D, which gives us the value $8.3\,\mathrm{m}$. If Viktor had not hidden in the passage, Jarda would still have only reached $20.6\,\mathrm{m}$, which is only $30\,\mathrm{cm}$ farther.

Jaroslav Herman
jardah@fykos.org

Problem 50 ... water in a bowl

7 points

We pour 650 ml *of water into a bowl shaped like a perfect homogenous hemisphere of radius* 12 cm *and mass* 610 g*. Then we start tilting the bowl. What work we must do to get the water to start pouring out? Assume that the water is an ideal liquid, and neglect the thickness of the walls.* *Jarda was rinsing a bowl which had had cucumber salad in it.*

Firstly, we note a few important observations. When we tilt the bowl, the water stays in place, so we do not change its potential energy. Furthermore, if we assume there is no friction between the water and the bowl, we are not doing any work at all because of the water. The only thing we are doing in the terms of the work is raising the center of gravity of the bowl itself.

Another interesting thing is the shape of the bowl. Since the thickness of the walls is small, we consider the bowl to be a hemisphere. Obviously, the bowl has some area density, which is a constant. Therefore, the mass is everywhere proportional to the surface area. Moreover, for a sphere, the area of the spherical segment is directly proportional to its height. From here, it simply follows that the center of gravity of the hemisphere is located at half of its height on the axis of symmetry, i.e. at $h_1 = R/2$, where $R = 12\,\mathrm{cm}$ denotes the radius of the bowl.

So far, we know the position of the center of gravity and that the work done is proportional to the change in its position. Now we need to calculate how much we need to tilt the bowl to get the water to start pouring out. That will occur when the water level is at the same height as the bottom of the bowl. Let us denote by α the angle that the normal to the water (and the ground) makes with the plane of the rim of the bowl.

At this moment, the height of the center of gravity above the ground

$$h_2 = R - \frac{R}{2}\sin\alpha\,.$$

It remains to obtain the value of the angle α from knowing the volume. These two quantities are bound by the equation[5]

$$V = \frac{\pi R^3}{3}\left(2+\cos\alpha\right)\left(1-\cos\alpha\right)^2\,.$$

That is a cubic equation for $\cos\alpha$. Its numerical solution is

$$\cos\alpha = 0.630\,47 \quad\Rightarrow\quad \alpha = 0.887\,28 \quad\Rightarrow\quad \sin\alpha = 0.776\,27\,.$$

The difference in potential energy of the bowl is thus

$$E_{\mathrm{p2}} - E_{\mathrm{p1}} = mg\left(h_2 - h_1\right) = mgR\frac{\left(1-\sin\alpha\right)}{2} = 80\,\mathrm{mJ}\,.$$

Jaroslav Herman
jardah@fykos.org

[5] See Wikipedia https://en.wikipedia.org/wiki/Spherical_cap.

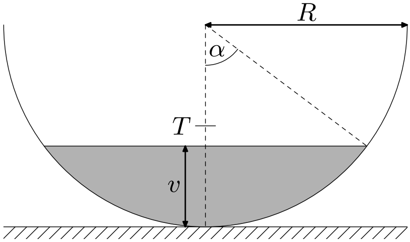


Fig. 3: Before tilting.

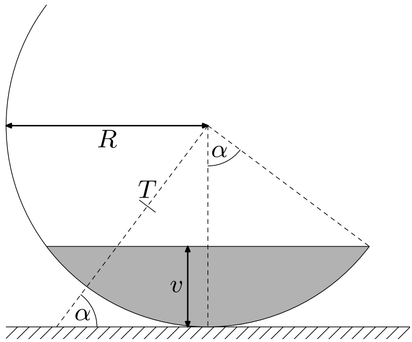


Fig. 4: When water pours out.

Problem 51 ... where to hide? 9 points

It is the year 2311 and aliens are about to attack Earth. They arrived in their spaceship and stopped at an altitude of $h = 12\,200\,\mathrm{km}$ *above the South Pole. Fortunately, the Earthlings had already developed a defensive missile system and destroyed the attackers' ship. It fell apart into thousands of tiny fragments, which flew in all directions at speeds up to a maximum of* $v = 300\,\mathrm{m{\cdot}s^{-1}}$*. What percentage of the Earth's surface do the fragments hit? The ship was made of refractory material, so the debris does not burn in the atmosphere.*

Jarda saw Anubis' fleet destroyed in Stargate.

The spaceship was shattered above Earth's pole, so we need not consider the Earth's rotation. It is clear that this problem is axisymmetric, so we will tackle the problem in two dimensions. Firstly we verify that the fragments have insufficient velocity to escape from the Earth's gravity by looking at the sign of the total mechanical energy per unit mass (denoted as E). We easily see that the value

$$E = \frac{v^2}{2} - \frac{\mu}{r_0} \doteq -21.4\,\mathrm{MJ{\cdot}kg^{-1}}$$

is negative, therefore all of the fragments have elliptical trajectories. The Earth's gravitational parameter, i.e. the product of Earth's density and gravitational constant, is denoted by $\mu \doteq 3.99 \cdot 10^{14}\,\mathrm{m^3{\cdot}s^{-2}}$, and $r_0 = R_\mathrm{Z} + h$ is the distance between the point of explosion and Earth's center.

Firstly we will find the region of space which can be hit by fragments. Then we will find the intersection of this set of points with the surface of the Earth. We consider only fragments with the maximal velocity (v). All of these fragments thus have the same total energy (sum of kinetic and potential energy) since their distance from Earth's center is the same. The total energy stays constant during motion and it corresponds with the major half-axis of ellipse a according to

$$E = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}\,.$$

Thus all trajectories of fragments have the same major half-axis

$$a = \frac{\mu r_0}{2\mu - r_0 v^2}\,.$$

We use the following reasoning to find the boundary of the space where fragments can sweep. We choose a ray p pointing from the Earth's center (i.e. focus F_1) in an arbitrary direction. We then try to find the farthest point on p, which still intersects a trajectory k of any fragment. Let us denote this point as B and its distance from Earth's center (first focus) as ρ. Let s be the line segment between this point and the second focus F_2 of a given trajectory k, while we still don't know the position of this focus. Certainly, it must hold $\rho + s = 2a$ since the point B lines on the ellipse k.

The second focus F_2 can be connected with the point of the explosion by a line segment with length t and it also holds $r_0 + t = 2a$. We already know the length of a, thus we know the length of the line segment t. The focus F_2 must lie somewhere on the circle k_1 with radius $2a - r_0$ and center in the point of the explosion.

The point B thus lies un the ray p and its distance from the focus F_1 is ρ. The focus F_2 must lie on the circle k_2 with radius $2a - \rho$ and center in the point B. The value of ρ should be the largest, so the radius k_2 should be the lowest. This focus also lies on the circle k_1. Both of these conditions are fulfilled iff the circles k_1 and k_2 have exactly one common point. This point is the focus F_2 and it lies on the line connecting the centers of both circles.

Distance of point B is then readily computed as a sum of both radii, thus $2a - r_0 + 2a - \rho$. The important fact is that the sum of distances from point B to the center of Earth and from point B to the point of the explosion (i.e $4a - r_0 - \rho$) is constant and equal to $4a - r_0 - \rho + \rho = 4a - r_0$ for each ray on which point B may lie. Therefore, the set of all such points is an ellipse with one focus in the center of Earth and the second focus in the center of the explosion. The major half-axis of the ellipse is

$$a_\mathrm{o} = \frac{4a - r_0}{2} = \frac{r_0}{2}\frac{2\mu + r_0 v^2}{2\mu - r_0 v^2} \doteq 9\,328\,\mathrm{km}\,.$$

The only remaining task is to find the intersection of this ellipse with Earth. Let's introduce the cartesian coordinate system with the origin at the center of the line segment between the point of the explosion and Earth's center, i.e. in the center of the envelope ellipse (the ellipse no fragment can pass through). We know the major half-axis of the envelope ellipse and we can calculate the minor half-axis

$$b_\mathrm{o} = \sqrt{a_\mathrm{o}^2 - e^2} = \sqrt{a_\mathrm{o}^2 - \frac{r_0^2}{4}} = \frac{r_0\sqrt{2\mu r_0 v^2}}{(2\mu - r_0 v^2)}\,.$$

The equation of envelope ellipse is thus

$$\frac{x^2}{a_\mathrm{o}^2} + \frac{y^2}{b_\mathrm{o}^2} = 1\,,$$

while the equation of circle (Earth's surface) is

$$\left(x + \frac{r_0}{2}\right)^2 + y^2 = R_\mathrm{Z}^2\,.$$

We plug y^2 from the second equation to the first one and then we solve a quadratic equation for x. After a series of adjustments we obtain the result

$$x_{1,2} = \frac{2a_\mathrm{o}}{r_0}\left(-a_\mathrm{o} \pm R_\mathrm{Z}\right) \quad \Rightarrow \quad x = \frac{2a_\mathrm{o}}{r_0}\left(-a_\mathrm{o} + R_\mathrm{Z}\right) \doteq -2\,962\,\mathrm{km}\,.$$

We choose the positive sign (negative sign does not make sense).

We compute how far are the points of intersection of envelope ellipse from the South Pole

$$l = \left|\frac{r_0}{2} + x - R_\mathrm{Z}\right| \doteq 51.4\,\mathrm{km}\,.$$

The area which can be hit by the fragments of the spaceship is a spherical canopy with height l. Hence the fragments will hit Úlomky tak bude zasaženo

$$\frac{2\pi R_\mathrm{Z} l}{4\pi R_\mathrm{Z}^2} \doteq 0.004\,0 = 0.40\,\%$$

of Earth's surface.

The correct way of solving this problem would employ the ellipsoidal shape of the Earth, which would make the calculations even more difficult. The polar radius of Earth is $R_\mathrm{p} = 6\,357\,\mathrm{km}$, while the equatorial radius is $R_\oplus = 6\,378\,\mathrm{km}$. The value of l is relatively small so we can consider an osculating sphere with the same curvature at the South Pole $R = R_\oplus^2/R_\mathrm{p} \doteq 6\,399\,\mathrm{km}$. But we cannot just simply plug this value into the above expressions, because the center of the osculating sphere is not at the same point as the center of Earth. Using this more accurate approach correctly we obtain a more accurate value of $0.398\,\%$.

Jaroslav Herman
jardah@fykos.org

Problem 52 ... reading against the light — 9 points

Imagine a large plane with an albedo (a measure of reflectivity) of 0.69. *At a height* $h = 1.0\,\mathrm{m}$ *above the plane, there is a point source of light illuminating it. Precisely below the source, we place a black disc with a radius h at a height x, parallel to the plane. Determine the value of x for which the center of the bottom face of the disc receives the greatest luminous flux. Assume that the reflected light is scattered uniformly in all directions and the plane has a Lambertian surface.* *Matěj wanted to read while lying down, but the only light source was the chandelier.*

First, realize that albedo is redundant information. Since we are only interested in the maximum, we don't have to worry about "how much light" reaches the center of the disk in absolute terms.

Before we get into the solution to the problem, we need to understand the properties of a point source. The essential characteristic of a point source is that it radiates power evenly in all directions. This can be interpreted as follows. If the total radiated power is equal to P, the measured intensity I at a distance r will be proportional to P/r^2 since all the power will be uniformly distributed over a spherical area $4\pi r^2$.

Furthermore, we will need Lambert's cosine law, which states that the luminous flux scattered by a perfectly opaque surface is directly proportional to the cosine of the scattering angle (measured from the perpendicular). We should also remember that the incident light flux (whether in the case of a plane or a black disk) is proportional to the cosine of the incident angle. Finally, since the problem is axially symmetric with respect to the line passing through the source and the center of the disk, it will be useful to divide the plane on which the light is scattered into concentric, infinitesimally thin rings centered at the point where the axis of symmetry intersects the plane. If we denote the radius ρ of one such ring of width $\mathrm{d}\rho$, its area will be equal to $2\pi\rho\,\mathrm{d}\rho$. Equipped with this knowledge, we can then say that the luminous flux delivered to the center of the bottom of the disk by the rays scattered by one infinitesimally thin ring of radius ρ will be

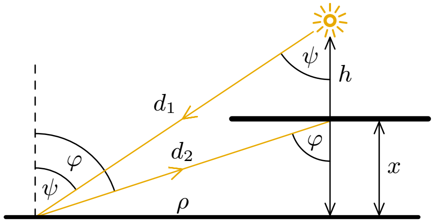


Fig. 5: Simple geometry.

- inversely proportional to the square of the distance between the source and the point of incidence on the plane (denoted by d_1),
- directly proportional to the cosine of the angle of incidence on this plane (let's denote ψ),
- directly proportional to the area of the ring $2\pi\rho\,\mathrm{d}\rho$,
- directly proportional (according to Lambert's law) to the cosine of the scattering angle towards the center of the disk (let's denote φ),
- inversely proportional to the square of the distance between the scattering point and the center of the black disc (let's denote d_2), and finally
- directly proportional to the cosine of the incidence angle to the target location (also equal to φ, from the alternation of angles).

By simple geometry, we can then express these quantities as

$$d_1 = \sqrt{h^2+\rho^2}\,, \quad \cos(\psi) = \frac{h}{d_1} = \frac{h}{\sqrt{h^2+\rho^2}}\,,$$
$$d_2 = \sqrt{x^2+\rho^2}\,, \quad \cos(\varphi) = \frac{x}{d_2} = \frac{x}{\sqrt{x^2+\rho^2}}\,.$$

Now we determine the minimum value of $\rho = \rho_0$ and integrate over all from ρ_0 to ∞. The value of ρ_0 corresponds to the radius of the shadow from the disc, which we determine from the similarity of the triangles as

$$\frac{\rho_0}{h} = \frac{h}{h-x} \quad \Rightarrow \quad \rho_0 = \frac{h^2}{h-x}\,.$$

The luminous flux at the center of the disk will therefore be proportional to the value of

$$hx^2 \int_{\frac{h^2}{h-x}}^{\infty} \frac{\rho}{(h^2+\rho^2)^{3/2}(x^2+\rho^2)^2}\,\mathrm{d}\rho\,.$$

We can solve this integral numerically. By plugging in Desmos, we can even plot the graph of this function, from which we can see that the maximum is $x \doteq 0.311\,\mathrm{m}$.

Vojtěch David
vojtech.david@fykos.org

Radka Křížová
radka.krizova@fykos.org

Problem 53 ... rattlesnake 11

8 points

Semir and his wingman are chasing a criminal in a police car. The perpetrator drove off the road onto a horizontal forest path, thinking he could outrun the highway cops. However, he did not notice that the road ended with a cliff, $H = 20\,\mathrm{m}$ deep. The criminal is approaching the cliff with a velocity $v = 144\,\mathrm{km{\cdot}h^{-1}}$. At what angle does the front bumper hit the ground? The parameters of the perpetrator's car are: mass $m = 1.32\,\mathrm{t}$, wheelbase $D = 2.46\,\mathrm{m}$, distance from the front axle to bumper $d_\mathrm{n} = 0.71\,\mathrm{m}$, the height of the center of mass of the car above the ground $h_\mathrm{t} = 42\,\mathrm{cm}$, the horizontal distance of the center of mass from the front axle $d_\mathrm{t} = 80\,\mathrm{cm}$, the height of the front bumper above the ground $h_\mathrm{n} = 15\,\mathrm{cm}$, the moment of inertia of the car with respect to the center of mass $I = 1.67 \cdot 10^3\,\mathrm{kg{\cdot}m^2}$. The car uses front wheel drive. Assume that the rear wheels roll without resistance. *Jindra watches German crime series.*

At the moment when the front axle crosses the edge of the cliff, the car loses support at the front and gravity causes rotation around the axis passing through the points of contact between the rear wheels and the ground.

The normal force F exerted by the ground on the rear axle may change depending on the angle of rotation of the car. Therefore, we'll work with the assumption that crossing the edge of the cliff itself takes such a short time ($t_1 = D/v \doteq 6.15 \cdot 10^{-2}\,\mathrm{s}$) that the car remains almost horizontal during that time. Then we can consider the force F to be constant. This force F, which imparts rotation to the car, acts at the rear axle, so the length of the moment arm is $x_\mathrm{t} = D - d_\mathrm{t} \doteq 1.64\,\mathrm{m}$.

The motion of the car is described by two equations. The first one describes acceleration in the vertical direction (the axis y points upwards), the second equation describes rotational motion (we define angular acceleration to be positive)

$$\begin{aligned} m\ddot{y} &= -mg + F, \\ I\dot{\omega} &= F x_\mathrm{t}. \end{aligned}$$

The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. We can express F from the second equation and substitute into the first equation to obtain

$$\ddot{y} = -g + \frac{I}{m x_\mathrm{t}} \dot{\omega}.$$

We know that the drop of the center of mass due to vertical forces must correspond to drop of the center of mass due to rotation around the rear axle, so

$$\dot{\omega} = -\frac{\ddot{y}}{x_\mathrm{t}}.$$

Both equations derived above can be combined into one, which lets us express the angular acceleration $\dot{\omega}$ using known quantities as

$$\dot{\omega} = \frac{g}{x_\mathrm{t}\left(1 + \frac{I}{m x_\mathrm{t}^2}\right)}.$$

Alternative approach: We would find the same angular acceleration in the reference frame of the rear axle, where gravity instead imparts rotation to the car and the moment of inertia can be found using Steiner's theorem.

This angular acceleration is only imparted onto the car during the time $t_1 = D/v$, while the rear wheels stay in contact with the ground. During this time, the car gains an angular velocity

$$\omega = \dot{\omega} t_1 = \frac{gD}{x_\mathrm{t} v \left(1 + \frac{I}{m x_\mathrm{t}^2}\right)} \doteq 0.25\,\mathrm{rad{\cdot}s^{-1}}$$

and since its initial angular velocity was zero, it rotates by an angle

$$\theta = \frac{1}{2}\dot{\omega} t_1^2 = \frac{gD^2}{2 x_\mathrm{t} v^2 \left(1 + \frac{I}{m x_\mathrm{t}^2}\right)} \doteq 7.7 \cdot 10^{-3}\,\mathrm{rad} \doteq 0.44^\circ .$$

Our assumption that the angle of rotation while crossing the cliff should be small turns out to be justified.

From the moment when the rear wheels also cross the edge of the cliff, the car is falling towards the ground along a parabolic trajectory and rotating with constant angular velocity $\omega = 0.25\,\mathrm{rad{\cdot}s^{-1}}$ around the rear axle. The center of mass would hit the ground after a time

$$t_\mathrm{t} = \sqrt{\frac{2H}{g}} \doteq 2.0\,\mathrm{s}\,,$$

where we neglected the depth of the front bumper under the rear axle at the moment of impact, since $D + d_\mathrm{n}$ is much smaller than H. At that moment, the angle of rotation of the car is approximately

$$\theta_\mathrm{n} \approx \omega t_\mathrm{t} \doteq 29^\circ\,,$$

where we can see that the contribution of the angle θ is also negligible.

Is this result sufficiently accurate? It's true that in reality, both the fall of the car and its rotation would be affected by air resistance, and therefore, more complex calculations still wouldn't correspond to the real world. However, if we didn't disregard anything except friction and air resistance, we could find the result 28.2° numerically.

Jindřich Jelínek
`jjelinek@fykos.org`

Problem 54 ... spring with friction — 8 points

Consider a massless spring with stiffness $k = 9.0\,\mathrm{N{\cdot}m^{-1}}$. *We attach a weight with mass* $m = 110\,\mathrm{g}$ *to its free end. The other end of the spring is anchored in a wall. The spring is horizontal and the weight slides on the floor with friction. The coefficient of friction is* $f = 0.35$, *and the rest length of the spring is* $l_0 = 30\,\mathrm{cm}$. *By pulling on the weight, the spring is stretched by* $x_0 = 20\,\mathrm{cm}$ *and then released with zero initial speed. When the weight comes to a complete stop, what will the length of the spring be?* *Jindra was playing with a slinky.*

The movement of the weight is described by the differential equation

$$m\ddot{x} = -kx - fmg\,\mathrm{sgn}(\dot{x})\,, \tag{7}$$

where g is the acceleration due to gravity and $\mathrm{sgn}(\dot{x})$ is the sign function

$$\mathrm{sgn}(\dot{x}) = \begin{cases} +1 & \dot{x} > 0\,, \\ 0 & \dot{x} = 0\,, \\ -1 & \dot{x} < 0\,. \end{cases}$$

The sign function ensures that the frictional force always acts against the direction of velocity.

Recall the basic property of the friction force, where at a non-zero velocity it has the magnitude fmg. In the static case, the friction force has a magnitude less than or equal to fmg, always so that the resultant of the forces acting on the body is zero. To set the weight in motion from rest, the applied force of the spring must be greater than fmg. The limiting case is

$$x_{\mathrm{max}} = \frac{fmg}{k} \doteq 4.2\,\mathrm{cm}\,.$$

If the weight stands still less than 4.2 cm from the equilibrium position, it will stay in place. The spring does not exert sufficient force to move the weight. For each swing, the direction of the friction force will change, so we will have to successively find a solution to the equation (7) with several different initial conditions. The equation describing the motion of the weight from release to reaching the extreme position is of the form

$$m\ddot{x} + kx = fmg$$

and the initial conditions are $x(0) = x_0 = 20\,\mathrm{cm}$ and $\dot{x}(0) = 0\,\mathrm{m{\cdot}s^{-1}}$. The general solution of this equation is

$$x(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{fmg}{k}\,,$$

where A and B are the integration constants and $\omega = \sqrt{k/m}$ we denote the angular frequency of the oscillations. The values of the constants A and B are found from the initial conditions $A = x_0 - fmg/k$ and $B = 0\,\mathrm{m}$. The body oscillates to the extreme position $x_1 = -x_0 + 2fmg/k$ at time $t_1 = \pi/\omega$. We quantify $x_1 \doteq -11.6\,\mathrm{cm}$ and see $|x_1| > x_{\mathrm{max}}$, which means the body starts moving in the opposite direction.

At that point the direction of velocity changes, so the motion is now described by the equation

$$m\ddot{x} + kx = -fmg$$

with initial conditions $x(t_1) = x_1$ and $\dot{x}(t_1) = 0\,\mathrm{m{\cdot}s^{-1}}$. The general solution is

$$x(t) = A\cos(\omega t) + B\sin(\omega t) - \frac{fmg}{k}\,,$$

where the integration constants are $A = -x_1 - fmg/k = -x_0 - 3fmg/k$ and $B = 0\,\mathrm{m}$. The body will oscillate to the extreme position $x_2 = x_0 - 4fmg/k$ at time $t_2 = 2\pi/\omega$ and at that instant its velocity will be zero. We quantify $x_2 \doteq 3.2\,\mathrm{cm}$ and see $|x_2| < x_{\mathrm{max}}$, so the body stops at that position.

The length of the spring after the weight stops will be $l = l_0 + x_2 \doteq 33.2\,\mathrm{cm}$.

Jindřich Jelínek
`jjelinek@fykos.org`

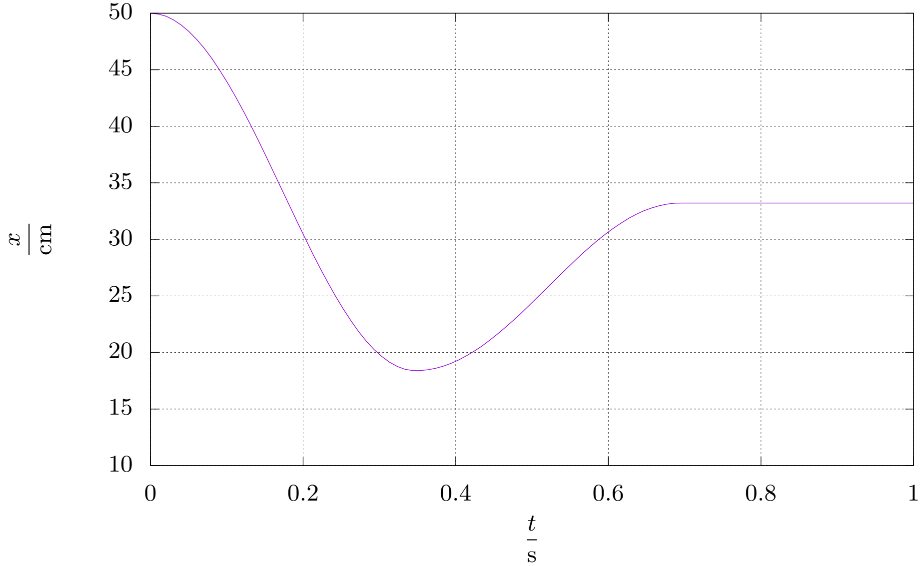


Fig. 6: Spring length vs. time.

Problem 55 ... central spring

9 points

Consider a spring attached at a point $r = 1\,321\,\mathrm{km}$ away from the center of a planet with mass $M = 4.11 \cdot 10^{22}\,\mathrm{kg}$ and radius $R = 1\,220\,\mathrm{km}$. The spring has a free length $x_0 = 89\,\mathrm{km}$ and a homogeneous linear density $\lambda = 8.7\,\mathrm{kg{\cdot}m^{-1}}$. What is the stiffness which the spring must have if we require it to exactly reach the surface of the planet? Do not consider any rotation of the system.

Jáchym was sorry that he had forgotten about a problem with a non-massless spring, so he decided to correct it.

Let us introduce a usual notation for heavy springs. A quantity x denotes the section of a length $\mathrm{d}x$ of the relaxed spring, whereas $y(x)$ is a coordinate of the given section on the stretched spring. We measure the distances from the point of attachment towards the center of the planet.

A gravitational force acting on the section $\mathrm{d}x$ is

$$\mathrm{d}F_\mathrm{g} = \frac{GM}{(r-y)^2} \lambda \,\mathrm{d}x\,.$$

Furthermore, the section $\mathrm{d}x$ is subject to an elastic (which we denote F) force emerging from adjacent sections. We can find this force by thinking of the section as a small individual spring with its stiffness $k_{\mathrm{d}x} = kx_0/\,\mathrm{d}x$, which extends to a length $\mathrm{d}y$. Therefore

$$F = k_{\mathrm{d}x}\left(\mathrm{d}y - \mathrm{d}x\right) = kx_0 \frac{\mathrm{d}y - \mathrm{d}x}{\mathrm{d}x} = kx_0\left(y' - 1\right)\,,$$

where k is the sought stiffness of the original spring.

At the beginning of the selected section, this force has a magnitude $F(x)$, at the end it would be $F(x+\mathrm{d}x)$. The difference between these forces is compensated by the gravitational force

$$F(x+\mathrm{d}x) - F(x) = -\,\mathrm{d}F_{\mathrm{g}}\,.$$

This happens to look similar when compared with the derivative definiton, so we can write

$$\frac{F(x+\mathrm{d}x)-F(x)}{\mathrm{d}x} = F' = -\frac{\mathrm{d}F_{\mathrm{g}}}{\mathrm{d}x} = -\frac{GM}{(r-y)^2}\lambda\,.$$

We've already had written F explicitly, so we just need to calculate its derivative and obtain

$$(r-y)^2\,y'' = -\frac{GM\lambda}{kx_0} \quad \Rightarrow \quad z^2 z'' = -\frac{GM\lambda}{kx_0} = -K\,, \tag{8}$$

where we have introduced a new coordinate $z = y - r$ and a constant K to simplify it. We use a common trick

$$z'' = z'\frac{\mathrm{d}z'}{\mathrm{d}z} \quad \Rightarrow \quad z'\,\mathrm{d}z' = -Kz^{-2}\,\mathrm{d}z \quad \Rightarrow \quad \frac{z'^2}{2} = Kz^{-1} + C\,.$$

Now is the right time to think about boundary conditions. Apparently $y(0) = 0$ and $y_0 = y(x_0) = r - R$. If we define $z_0 = z(y_0) = y_0 - r$, $z_0 = -R$ holds true. Furthermore, at point y_0 the spring is no longer tensioned, so we can write $y'(x_0) = 0$. Then, $z' = y'$ and $z'(y_0) = y'(x_0) = 0$. We can determine the constant of integration from

$$0 = Kz_0^{-1} + C \quad \Rightarrow \quad C = -Kz_0^{-1} \quad \Rightarrow \quad z' = \sqrt{2K}\sqrt{z^{-1} - z_0^{-1}}\,.$$

Notice that z is in a range $z(0) = y(0) - r = -r$ to $z_0 = -R$. After reasoning, we can tell that $y' > 0$, so even $z' > 0$, therefore, z is an ever-increasing negative function. We verified that the square root is well defined and we can proceed further. We separate the equation again and rewrite in an integral form

$$\sqrt{2K}\int \mathrm{d}x = \int \left(z^{-1} - z_0^{-1}\right)^{-\frac{1}{2}}\mathrm{d}z = \int \sqrt{\frac{z}{1-\frac{z}{z_0}}}\,\mathrm{d}z = z_0\sqrt{-z_0}\int\sqrt{\frac{\zeta}{\zeta-1}}\,\mathrm{d}\zeta\,,$$

where we used a substitution $\zeta = z/z_0$ and factored the terms so that the argument of square root $\sqrt{-z_0}$ is positive. There is no nice antiderivative of this function; however, we do not need it to solve the problem. We figure out the limits of integration as we integrate from the beginning till the end of spring. Now, we can solve the integral numerically

$$I_0^{x_0} = \int_{\frac{r}{R}}^{1}\sqrt{\frac{\zeta}{\zeta-1}}\,\mathrm{d}\zeta \doteq -0.583\,.$$

To express K, we also need to integrate the other side of the equation

$$z_0\sqrt{-z_0}I_0^{x_0} = \sqrt{2K}\int_0^{x_0}\mathrm{d}x = \sqrt{2K}x_0\,,$$

$$K = \frac{1}{2}\left(\frac{z_0\sqrt{-z_0}I_0^{x_0}}{x_0}\right)^2 = \frac{1}{2}\left(\frac{-R^{\frac{3}{2}}I_0^{x_0}}{x_0}\right)^2 = \frac{R^3}{2}\left(\frac{I_0^{x_0}}{x_0}\right)^2\,.$$

Finally, we plug the sought stiffness into the expression for K (8) and the result is

$$k = \frac{GM\lambda}{Kx_0} = \frac{2GM\lambda x_0}{\left(I_0^{x_0}\right)^2 R^3} \doteq 6.9\,\mathrm{N{\cdot}m^{-1}}\,.$$

Jáchym Bártík
tuaki@fykos.org

Problem 56 ... foggy glass 9 points

Consider horizontal glass with thickness $d_\mathrm{g} = 16.3\,\mathrm{\mu m}$ *and refractive index* $n_\mathrm{g} = 1.68$. *We spill water with refractive index* $n_\mathrm{w} = 1.33$ *on the glass in such a way that it forms a layer with uniform thickness* $d_\mathrm{w} = 15.5\,\mathrm{\mu m}$. *From the bottom, we shine a vertical beam of light with wavelength* $\lambda = 0.590\,\mathrm{\mu m}$ *on the glass. What is the ratio of the intensity of light which passes through in this situation to the intensity of light which would pass through if there was no water? Assume that light is only reflected or transmitted by the materials, not absorbed.*

Jarda's window got foggy.

When a plane electromagnetic wave hits an interface, reflections and passage of waves occur. Since there are multiple interfaces in the problem, there will be several internal reflections and interference. The reflection and transmission coefficients depend on the angles of the wave direction relative to the interfaces, the mediums' refractive indexes, and the polarization of the incident light. In the case of perpendicular incidence, the Fresnel equations for the transmission coefficients $t_{s,p}$ for both polarizations have the same form. For the reflection coefficients, the equation $r_\mathrm{s} = -r_\mathrm{p}$ holds, thus, the two coefficients differ only in sign. These relations, which depend on the refractive indices of the two media, tell us what the intensity of the reflected and transmitted radiation will be.

Since we have three interfaces in the problem, it is better to work with the electric intensity E rather than directly with I. We first solve the problem for one of the polarizations (e.g., for s-type) and then assign a negative sign to all r-reflection coefficients to obtain an expression for the p-type polarization. Thus, for a perpendicular incidence, the reflection coefficient at the interface for the s-polarization is equal to

$$r = \frac{E_r}{E_\mathrm{i}} = \frac{n_1 - n_2}{n_1 + n_2}\,,$$

where E_i is the vector of the electric intensity of the incident wave and E_r of the reflected wave. The refractive index n_1 belongs to the medium where the incident wave travels, and n_2 is the refractive index behind the interface. In our case, we have three media whose refractive indexes are $n = 1$ for air, $n_g = 1.5$ for glass, and $n_w = 1.33$ for water. We can see that if the wave is reflected from a medium with a higher refractive index, there is a phase change due to the coefficient being negative.

The transmission coefficient t is established analogically and has the following form

$$t = \frac{E_\mathrm{t}}{E_\mathrm{i}} = \frac{2n_1}{n_1 + n_2}\,,$$

where E_t is the transmitted electric intensity.

Radiation passes from one medium to another, but part of it gets reflected. Some of it then reaches the other interface, where a certain part of the radiation passes through again and some is reflected. This reflected portion returns to the first interface, where, of course, another passage and reflection occurs. It is evident that a simple estimation of the amount of total radiation that passes through is impossible. Iterating these considerations, we could arrive at a summation of infinite series. This method can be used for plane-parallel plates (2 interfaces), but in our case, it is no longer wise. Let us look at it differently.

In the next section, we introduce the notation for each value of intensity. However, the phase of these intensities changes as they pass through the environment. For this, we will use complex labeling. We will subscript the intensities in each environment and put their phase at the point of origin equal to zero. The change in phase depends on the thickness of the glass and water as well as on their refractive indexes.

Let us denote by E_i an incident wave for which (contrary to the definitions in the previous paragraph) we set the phase at impact as zero. The wave with direction from the first medium to the second is E_{s1}, and the one in the opposite direction is E_{s2}. The wave going from the first interface to space is E_r. It is important to note that in our notation we will not be dealing only with a reflected wave, but also with parts of waves that have been reflected at other interfaces.

The wave going from the second interface to the third is E_{v1}, in the opposite direction E_{v2}. The wave passing through the entire double layer is E_t. Again, we must remember that these are the sum of all the partial waves that are formed by the infinite reflections at the interfaces.

Let us denote the coefficient of passage from air to glass t_{ag}, from glass to water t_{gw}, from water to air t_{wa}, from glass to air t_{ga}, and from water to glass t_{wg}. Then

$$t_\mathrm{ag} = \frac{2}{1+n_\mathrm{g}}, \quad t_\mathrm{gw} = \frac{2n_\mathrm{g}}{n_\mathrm{g}+n_\mathrm{w}}, \quad t_\mathrm{wa} = \frac{2n_\mathrm{w}}{n_\mathrm{w}+1}, \quad t_\mathrm{ga} = \frac{2n_\mathrm{g}}{n_\mathrm{g}+1}, \quad t_\mathrm{wg} = \frac{2n_\mathrm{w}}{n_\mathrm{g}+n_\mathrm{w}}\,.$$

Similarly, let us denote the reflection coefficient between air and glass r_ag, between glass and water r_gw, between water and air r_wa, between glass and air r_ga, and between water and glass r_wg (the first environment is the one in which the incident wave is at the interface). Then

$$r_\mathrm{ag} = \frac{1-n_g}{1+n_g} = -r_\mathrm{ga}, \quad r_\mathrm{gw} = \frac{n_g-n_w}{n_g+n_w} = -r_\mathrm{wg}, \quad r_\mathrm{wa} = \frac{n_w-1}{n_w+1}\,.$$

Let us now denote the phase shift between the air-glass and glass-water interface by δ_g which is equal to

$$\delta_\mathrm{g} = \frac{2\pi n_\mathrm{g} d_\mathrm{g}}{\lambda}\,,$$

where d_g is the thickness of the glass and *lambda* is the wavelength of light. By analogy, we introduce the phase shift δ_w in water.

We see that we must distinguish the order of the environments. Now we can start building relationships between the different electrical intensities. For E_g1

$$E_\mathrm{g1} = t_{ag}E_\mathrm{i} + r_{ga}E_\mathrm{g2}\mathrm{e}^{i\delta_\mathrm{g}}\,,$$

is the sum of the transmitted and reflected waves multiplied by their respective coefficients. The reflected wave E_g2 must be multiplied by the change in its phase. Then we have

$$E_\mathrm{g2} = r_\mathrm{gw}E_\mathrm{g1}\mathrm{e}^{i\delta_\mathrm{g}} + t_\mathrm{wg}E_\mathrm{w2}\mathrm{e}^{i\delta_\mathrm{w}}, \quad E_r = r_\mathrm{ag}E_\mathrm{i} + t_\mathrm{ga}E_\mathrm{g2}\mathrm{e}^{i\delta_\mathrm{g}}\,.$$

For waves in water

$$E_\mathrm{w1} = t_\mathrm{gw}E_\mathrm{g1}\mathrm{e}^{i\delta_\mathrm{g}} + r_\mathrm{wg}E_\mathrm{w2}\mathrm{e}^{i\delta_\mathrm{w}}, \quad E_\mathrm{w2} = r_\mathrm{wa}E_\mathrm{w1}\mathrm{e}^{i\delta_\mathrm{w}}, E_\mathrm{t} = t_\mathrm{wa}E_\mathrm{w1}\mathrm{e}^{i\delta_\mathrm{w}}\,.$$

If we consider that E_i is known, then we have just obtained a system of six equations with six unknowns. Our task is to express E_t in terms of E_i.

We have

$$E_\mathrm{t} = t_\mathrm{wa} E_\mathrm{w1} \mathrm{e}^{i\delta_\mathrm{w}} = t_\mathrm{wa} t_\mathrm{gw} \frac{E_\mathrm{g1} \mathrm{e}^{i\delta_\mathrm{w} + i\delta_\mathrm{g}}}{1 - r_\mathrm{wg} r_\mathrm{wa} \mathrm{e}^{i2\delta_\mathrm{w}}}$$

and

$$E_\mathrm{g1} - r_\mathrm{ga} \mathrm{e}^{i\delta_\mathrm{g}} \left(r_\mathrm{gw} \mathrm{e}^{i\delta_\mathrm{g}} + t_\mathrm{wg} r_\mathrm{wa} t_\mathrm{gw} \frac{\mathrm{e}^{i\delta_\mathrm{w} + i\delta_\mathrm{g}}}{1 - r_\mathrm{wg} r_\mathrm{wa} \mathrm{e}^{i2\delta_\mathrm{w}}} \right) E_\mathrm{g1} = t_\mathrm{ag} E_\mathrm{i} \,,$$

from where

$$E_\mathrm{t} = \frac{t_\mathrm{wa} t_\mathrm{gw} t_\mathrm{ag} \mathrm{e}^{i\delta_\mathrm{g} + i\delta_\mathrm{w}}}{\left(1 - r_\mathrm{wg} r_\mathrm{wa} \mathrm{e}^{i2\delta_\mathrm{w}}\right)\left(1 - r_\mathrm{ga} r_\mathrm{gw} \mathrm{e}^{i2\delta_\mathrm{g}}\right) - r_\mathrm{ga} t_\mathrm{wg} r_\mathrm{wa} t_\mathrm{gw} \mathrm{e}^{i2\delta_\mathrm{g} + i2\delta_\mathrm{w}}} E_\mathrm{i} \,.$$

We've got a symmetric expression, which is good news. In addition, the reflection coefficients r always occur as couples in products. Thus, if we replace all these numbers with negative ones (when we change the polarization from s to p), the result does not change because the negative signs are multiplied to positive ones. From this point on, we no longer need to distinguish between s and p polarizations. We now must calculate the square of the absolute value. The complex expression in the numerator will not play a role. However, we must multiply the denominator by its complex conjugate number.

In the denominator we substitute $a = r_\mathrm{wg} r_\mathrm{wa}$, $b = r_\mathrm{ga} r_\mathrm{gw}$ and $c = r_\mathrm{ga} t_\mathrm{wg} r_\mathrm{wa} t_\mathrm{gw}$. We then expand it and multiply the whole by its complex conjugated value. After the adjustment, we get

$$\begin{aligned} &1 + a^2 + b^2 + a^2 b^2 + c^2 \\ &-2\left(a\left(1 + b^2 - \frac{bc}{a}\right)\cos 2\delta_\mathrm{w} + b\left(1 + a^2 - \frac{ac}{b}\right)\cos 2\delta_\mathrm{g}\right) \\ &-2\left((c - ab)\cos(2\delta_\mathrm{g} + 2\delta_\mathrm{w}) - ab\cos(2\delta_\mathrm{w} - 2\delta_\mathrm{g}) + abc\right) \,. \end{aligned}$$

Now consider that $n_\mathrm{w} = n_\mathrm{a} = 1$, meaning that the air is straight behind the glass. Then, after plugging for a, b and c we have $a = 0$, $b = r_\mathrm{ga} r_\mathrm{ga} = \left(\frac{1 - n_g}{1 + n_g}\right)^2$ and $c = 0$. The square of the absolute value of the electric intensity is then

$$I_{tg} = \frac{\left(t_\mathrm{ga} t_\mathrm{ag}\right)^2}{1 + \left(r_\mathrm{ga} r_\mathrm{ga}\right)^2 - 2 r_\mathrm{ga} r_\mathrm{ga} \cos 2\delta_\mathrm{g}} I_i \,,$$

where in the numerator we no longer have $(t_\mathrm{wa})\, ani\, (t_\mathrm{gw})$, since there is no water.

By comparison we get the final result (in which we have neglected some small terms in the denominator)

$$\frac{I_\mathrm{tgw}}{I_\mathrm{tg}} = \frac{t_\mathrm{wa}^2 t_\mathrm{gw}^2 t_\mathrm{ga}^{-2}\left(1 + (r_\mathrm{ga} r_\mathrm{ga})^2 - 2 r_\mathrm{ga} r_\mathrm{ga} \cos 2\delta_\mathrm{g}\right)}{1 + a^2 + b^2 + c^2 - 2\left(a\left(1 - \frac{bc}{a}\right)\cos 2\delta_\mathrm{w} + b\left(1 - \frac{ac}{b}\right)\cos 2\delta_\mathrm{g} + (c - ab)\cos(2\delta_\mathrm{g} + 2\delta_\mathrm{w}) - ab\cos(2\delta_\mathrm{w} - 2\delta_\mathrm{g})\right)} \,,$$

$$\frac{I_\mathrm{tgw}}{I_{tg}} = 0.96 \,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem 57 ... Matějʼs black warm box

7 points

Determine the number of photons in a sealed box of exact volume $1\,\ell$*, which is in thermal equilibrium in a room with a temperature of* $25.0\,\mathrm{K}$.

Karel was wondering what the head of Physics Brawl Online does when out of the public eye.

To determine the number of photons in the box, we need to find the number density. It could seem we only need to use the Stefan–Boltzmann law to determine the energy density, but it's not that simple because we don't know the mean energy of a single photon. We must, therefore, proceed from the first principles – Planck's law

$$B(T,\nu) = h\nu \frac{2\nu^2}{c^2} \frac{1}{\mathrm{e}^{\frac{h\nu}{k_\mathrm{B} T}} - 1}\,,$$

where ν is the frequency of radiaton, T is the absolute temperature of black body, h is the Planck constant, c is the speed of light and k_B is the Boltzmann constant. The spectral radiance of a body B describes the spectral emissive power per unit area, per unit solid angle for particular radiation frequencies from $[\nu, \nu + \mathrm{d}\nu]$ frequency interval. Hence the energy density is

$$u = \frac{4\pi}{c} L = \frac{4\pi}{c} \int_0^\infty B(T,\nu)\,\mathrm{d}\nu\,.$$

The result of this integration is the Stefan–Boltzmann law. However, our goal is to determine the number density of photons, so we divide the integrand $B(T,\nu)$ by energy of a single photon $h\nu$

$$n = \frac{4\pi}{c} \int_0^\infty \frac{B(T,\nu)}{h\nu}\,\mathrm{d}\nu = \frac{4\pi}{c} \int_0^\infty \frac{2\nu^2}{c^2} \frac{1}{\mathrm{e}^{\frac{h\nu}{k_\mathrm{B} T}} - 1}\,\mathrm{d}\nu\,.$$

The next step is to convert this integral containing physical quantities into a mathematical integral (to get a dimensionless integrand by a substitution – a frequently used method of converting a computational physical problem to a mathematical one). The most important is to have a simple variable in the argument of the exponential function, therefore, we use

$$\frac{h\nu}{k_\mathrm{B} T} = x \quad \Rightarrow \quad \nu = \frac{x k_\mathrm{B} T}{h}\,.$$

After this substitution we get

$$n = \frac{4\pi}{c} \int_0^\infty \frac{2\left(\frac{x k_\mathrm{B} T}{h}\right)^2}{c^2} \frac{1}{\mathrm{e}^x - 1} \frac{k_\mathrm{B} T}{h}\,\mathrm{d}x = 8\pi \left(\frac{k_\mathrm{B} T}{ch}\right)^3 \int_0^\infty \frac{x^2}{\mathrm{e}^x - 1}\,\mathrm{d}x\,.$$

There would be x to the third power in the numerator in case of determining the energy. This integral can be solved numerically, or by integral definition of the Riemann zeta function

$$\int_0^\infty \frac{x^2}{\mathrm{e}^x - 1}\,\mathrm{d}x = \zeta(3)\,\Gamma(3) = 2\zeta(3) \doteq 2.404\,.$$

The final equation we use to get the answer is

$$N = Vn = 2.404 \cdot 8\pi \left(\frac{k_\mathrm{B} T}{ch}\right)^3 \cdot V \doteq 3.170 \cdot 10^8\,.$$

Jozef Lipták
liptak.j@fykos.org

Problem M.1 ... fast and even faster for the first time 3 points

Two cars travel the same distance s. Both are moving with constant acceleration. The first moves with acceleration a_1, and the second with acceleration $a_2 = 1.25\,a_1$. The initial velocity of both cars is zero. How fast will the second car travel the required distance? Enter the result into the system as a ratio of the second car's time to the first car's time.

Karel varied the problems.

We will start with the known equation for accelerated motion, which is very similar for both cars

$$s = \frac{1}{2}a_1 t_1^2 = \frac{1}{2}a_2 t_2^2\,,$$

where t_1 and t_2 is the time it takes the first and the second car, respectively to travel the distance s. This way we immediately obtained an equation from which we express t_2/t_1

$$\frac{t_2}{t_1} = \sqrt{\frac{a_1}{a_2}} = \sqrt{\frac{1}{1.25}} \doteq 0.894\,.$$

The correct answer is that the second car travels the same distance 0.894 times faster than the first one. We can also note that even if we change the acceleration of the second car by a quarter, the time will shorten just by 0.106 times the time it took the first car.

Karel Kolář
karel@fykos.org

Problem M.2 ... fast and even faster for the second time 3 points

We have two cars, and we want them to travel the distance s in the same time t. During their journey, they will both have constant acceleration in the direction of travel. The first car will travel with an acceleration of a_1, whereas the second car will have an acceleration of $a_2 = 1.250\,a_1$. The second car starts its movement from the rest. With what initial velocity v_0 must the first car start? Give the result as a multiple of $a_1 t$ (enter only the number by which you multiply these quantities).

Karel varied the problems for the second time.

Both cars have to travel the same distance, so

$$s = \frac{1}{2}a_1 t^2 + v_0 t = \frac{1}{2}a_2 t^2\,.$$

From the equation, we can simply express the velocity

$$v_0 t = \frac{1}{2}a_2 t^2 - \frac{1}{2}a_1 t^2 = \frac{1}{2}\left(a_2 - a_1\right) t^2\,,$$
$$v_0 = \frac{1}{2}\left(a_2 - a_1\right) t = \frac{1}{2}\cdot 0.250\,a_1 t = 0.125\,a_1 t\,.$$

The initial velocity expressed using the acceleration of the first car and total time is $v_0 = 0.125\,a_1 t$.

Karel Kolář
karel@fykos.org

Problem M.3 ... fast and even faster for the third time — 3 points

Two cars have to travel the same distance s. Both will be moving with the constant jerk, which is a change in acceleration over time (analogous to acceleration being a change in velocity). The first car will travel with jerk g_1 and the second with jerk $g_2 = 1.25\, g_1$. The initial velocities and accelerations of both cars are zero. How fast will the second car travel the required distance? Enter the result into the system as the ratio of the second car's time to the first.

Karel varied the problems for the third time.

This is a simple application of the jerk if we know the formula for position versus time (which can be found on the internet, but with a little more effort) or we derive it by integration easily. Knowing that the jerk is constant, the following holds for acceleration

$$a_i = \int_0^{t_i} g_i \,\mathrm{d}\tilde{t} = g_i t_i \,,$$

where i can be 1 or 2 – depending on whether we want the formula for the first or the second car, and a is acceleration. We proceed with the integration for the velocity v and the position s, remembering that we have initial conditions for velocity and acceleration equal to zero.

$$v_i = \int_0^{t_i} a_i \,\mathrm{d}\tilde{t} = \int_0^{t_i} g_i \tilde{t} \,\mathrm{d}\tilde{t} = \frac{1}{2} g_i t_i^2 \,,$$

$$s = \int_0^{t_i} v_i \,\mathrm{d}\tilde{t} = \int_0^{t_i} \frac{1}{2} g_i \tilde{t}^2 \,\mathrm{d}\tilde{t} = \frac{1}{6} g_i t_i^3 \,.$$

The last equation can be rewritten as an equation for both cars, and very quickly, we get to the result

$$\frac{1}{6} g_1 t_1^3 = \frac{1}{6} g_2 t_2^3 \qquad \Rightarrow \qquad \frac{t_2}{t_1} = \sqrt[3]{\frac{g_1}{g_2}} = \sqrt[3]{\frac{1}{1.25}} \doteq 0.928 \,.$$

The second car will finish in a time 0.928 times shorter than the first one.

We could reach the same result by logical reasoning about the analogy with accelerated motion. For motion with constant acceleration, the acceleration does not depend on time, the velocity depends on time linearly and the position quadratically. If we have a constant jerk, we will be adding powers of the time again. The acceleration will depend on time linearly, the velocity quadratically, and the position cubically. This simplified reasoning will not give us the factor 1/6, but we do not need to know this factor if we only want the ratio of the times of the two cars as the result.

Karel Kolář
`karel@fykos.org`

Problem M.4 ... fast and even faster for the fourth time — 3 points

Two cars have to travel the same distance s. Both will move with constant power. The power of the first car will be P_1, and the power of the second car will be $P_2 = 1.25\, P_1$. The initial velocity of both cars is zero, and they have the same mass, including the load. How fast will the second car travel the required distance? Enter the result into the system as the ratio of the second car's time to the first.

Karel varied the problems for the fourth time.

We start with the formula for the kinetic energy $E_{\mathrm{k},i}$ of the motion of the ith car

$$E_{\mathrm{k},i} = \frac{1}{2} m v_i^2 = P_i t_i \,,$$

where m is the mass (without the index because it is the same for both cars), v_i is the velocity of ith car, and t_i is the travel time. If we express the velocity, we get its dependence on time and other parameters, which we consider to be constant with time

$$v_i = \sqrt{\frac{2P_i t_i}{m}}\,.$$

However, we need to know the time dependence of the distance traveled. Therefore, we integrate the formula for the velocity

$$s = \int_0^{t_i} v_i \,\mathrm{d}\tilde{t} = \int_0^{t_i} \sqrt{\frac{2P_i \tilde{t}_i}{m}} \,\mathrm{d}\tilde{t} = \frac{2}{3}\sqrt{\frac{2P_i t_i^3}{m}}\,.$$

The distance traveled is the same for both cars, so we set up the following equation

$$\frac{2}{3}\sqrt{\frac{2P_1 t_1^3}{m}} = \frac{2}{3}\sqrt{\frac{2P_2 t_2^3}{m}}\,,$$

$$P_1 t_1^3 = P_2 t_2^3\,,$$

$$\frac{t_2}{t_1} = \sqrt[3]{\frac{P_1}{P_2}} \doteq 0.928\,.$$

The second car will travel the distance in 0.928 times the first car's time when having 25 % more power. Coincidentally, this is the same ratio of times as in the "fast and even faster for the third time" problem, where you were comparing two cars with a constant jerk. This holds even though the time dependencies are different.

Karel Kolář
`karel@fykos.org`

Problem E.1 ... light circle

4 points

A thin convex lens is placed at a distance of 7.0 cm *from the screen. We then turn on a distant light source located on its optical axis. An illuminated circle appears on the screen behind the lens. After moving the lens to a distance of* 9.0 cm *from the screen, we notice that the circle on the screen now has the same radius as before. What is the focal length of the lens?*

Jarda saw the opposite of the eclipse.

While the source is far from the lens, all the rays passing through the lens converge at its focus. They form a cone after passing through the lens. If the screen is between the lens and its focus, the intersection of the cone and screen forms a circle. In this case, let the distance between the focus and the screen be x_1. However, a circle of the same radius will appear when the screen is behind the focus and the rays passing through the focus begin to diverge again. Now let there be a distance x_2 between the screen and the focus. Since the radii are equal, $x_1 = x_2$.

Moreover, in the first case $x_1 = f - d_1$, where $d_1 = 7\,\mathrm{cm}$, and in the second case $x_2 = d_2 - f$ (where $d_2 = 9\,\mathrm{cm}$). Since these distances are equal, we obtain

$$f = \frac{d_1 + d_2}{2} = 8\,\mathrm{cm}\,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem E.2 ... planparallel lens

3 points

A thin convex lens with radii of curvature $R_1 = 20\,\mathrm{cm}$ and $R_2 = 35\,\mathrm{cm}$ is cut perpendicular to its optical axis. The two halves are placed with the convex sides close to one another (the flat surfaces are facing away from each other). What is the ratio between the old and the new focal lengths of this arrangement of glasses? The refractive index of the glass is $n = 1.5$.

According to Jarda, shape does not matter.

Let's look at the passage of a beam through two lenses that are next to each other. Let a be the distance of an object from the first lens with focal length f_1. Just behind it is the second lens with a focal length f_2. Using the thin lens equation, we find the position of the image a' of the object as

$$a' = \frac{af_1}{a - f_1}\,.$$

This distance is positive if the object is imaged behind the first lens. We project the image through the second lens, with the object distance now equal to $-a'$. The second lens displays the object at

$$a'' = \frac{-a' f_2}{-a' - f_2} = \frac{-\left(\frac{af_1}{a-f_1}\right) f_2}{-\frac{af_1}{a-f_1} - f_2} = \frac{af_1 f_2}{af_1 + af_2 - f_1 f_2} = \frac{aF}{a - F}\,,$$

where $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$. Thus, we derived the interesting relation that the total optical power of two lenses that are close to each other is equal to the sum of the optical powers of the two lenses.

Now note that neither f_1 nor f_2 depends on the rotation of the lens. It is also important for this problem that we have derived that F does not depend on the order of the lenses. Thus, in both cases from the problem statement, F is the same and is equal to the focal length of the original lens. Thus, in both cases from the problem statement, F is the same and equal to the original lens's focal length. The ratio we were looking for is, therefore 1.

Jaroslav Herman
`jardah@fykos.org`

Problem E.3 ... two magnifiers

4 points

Jarda found two magnifying glasses at home and started playing with them. When he put them close together, he discovered they had a focal length of $F = 7\,\mathrm{cm}$. He then placed the lenses at a suitable distance apart, stood far behind them, and looked at a distant tree. He saw it sharply and its magnitude was 1.5-times larger than without the lenses. What is the difference *in the focal lengths of the two lenses? Subtract the smaller value from the larger. Think of the magnifying glass as a thin lens.*

Jarda lives so high that he can't really see the ground from the window.

In the previous problem, we derived that for two lenses that are placed closely together, applies $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$, where f_1 and f_2 are the focal lengths of the two lenses and F is the focal length of the system we know from the problem.

Conversely, if Jarda looked at a distant tree through two lenses and saw it sharply, he built a Kepler telescope. For him, the foci of the two lenses are at the same point, so the lenses are $f_1 + f_2$ apart. The magnification of such a telescope is then $Z = \frac{f_1}{f_2}$ for $f_1 > f_2$.

From the two equations with two unknowns, we can calculate f_1 and f_2 as

$$f_1 = F\left(1 + Z\right) ,$$
$$f_2 = F\left(1 + \frac{1}{Z}\right) .$$

Since $Z > 1$, then $f_1 > f_2$, so we are interested in the result $f_1 - f_2$, which is

$$f_1 - f_2 = F\left(Z - \frac{1}{Z}\right) = 5.8\,\mathrm{cm}\,.$$

Jaroslav Herman
jardah@fykos.org

Problem E.4 ... recording ants 8 points

We place two lenses and a screen on an optical axis. One lens, with focal length $f_1 = 3.6\,\mathrm{cm}$, is located $d = 8.3\,\mathrm{cm}$ in front of the screen. The second lens with focal length $f_2 = -1.6\,\mathrm{cm}$ is located between the first lens and the screen. An ant is initially located $a_0 = 9.7\,\mathrm{cm}$ in front of the first lens and is walking along the optical axis towards the lenses and the screen with velocity $v = 0.6\,\mathrm{cm{\cdot}s^{-1}}$. The image of the ant on the screen is initially in focus. What should be the velocity of the second lens if we want the image to stay in focus for the next moment during the walk? The sign of this velocity should be positive if the lens is moving in the same direction as the ant.

Jarda likes to take photos of insects.

Let's denote the position of the ant by $a = a_0 - vt$. The position of the focused image formed by the first lens is

$$a' = \frac{af_1}{a - f_1}\,.$$

The distance of this image from the second lens is

$$a'' = d - x - a'\,,$$

where $x = x_0 + ut$ and x_0 is the distance of the second lens from the screen at the time $t = 0\,\mathrm{s}$.

The final focused image is formed by the second lens when we treat the image formed by the first lens as the object. The distance of the final image from the second lens is

$$a''' = \frac{a''f_2}{a'' - f_2} = \frac{(d - x - a')\,f_2}{d - x - a' - f_2} = x\,,$$

where the last equality describes that we see the ant on the screen in focus. We solve this quadratic equation to find

$$x = \frac{-a' + d \pm \sqrt{a' - d}\sqrt{a' - d + 4f_2}}{2}\,.$$

Then, we find

$$x_0 = \frac{1}{2}\left(-\frac{a_0 f_1}{a_0 - f_1} + d + \sqrt{\frac{a_0 f_1}{a_0 - f_1} - d}\sqrt{\frac{a_0 f_1}{a_0 - f_1} - d + 4f_2}\right) \doteq 3.69\,\mathrm{cm}\,,$$

where we chose the positive root according to the problem statement. Let's not fear the expressions under the square roots - they're both negative, so if we put them under a common square root, we end up taking the square root of a positive number.

Next, let's plug time-dependence into the formula for x. We're only dealing with the next moment of the walk, so we consider a very small value of t, for which $a_0 \gg vt$. Then

$$\begin{aligned} a' &= \frac{(a_0 - vt)\, f_1}{a_0 - vt - f_1} = \frac{(a_0 - vt)\, f_1}{\left(1 - \frac{vt}{a_0 - f_1}\right)(a_0 - f_1)} \\ &\approx \frac{(a_0 - vt)\, f_1}{(a_0 - f_1)}\left(1 + \frac{vt}{a_0 - f_1}\right) \approx \frac{a_0 f_1}{(a_0 - f_1)} + \frac{v f_1^2}{(a_0 - f_1)^2} t\,. \end{aligned}$$

For clarity, let's denote $b' = a'(t = 0\,\mathrm{s}) = \frac{a_0 f_1}{(a_0 - f_1)}$. Now we find the approximation

$$\sqrt{a' - d} = \sqrt{b' - d}\sqrt{1 + \frac{1}{b' - d}\frac{vb'^2}{a_0^2} t} \approx \sqrt{b' - d}\left(1 + \frac{1}{b' - d}\frac{vb'^2}{2a_0^2} t\right)\,.$$

Similarly, we find the approximation for the second square root

$$\begin{aligned} \sqrt{a' - d + 4f_2} &= \sqrt{b' - d + 4f_2}\sqrt{1 + \frac{1}{b' - d + 4f_2}\frac{vb'^2}{a_0^2} t} \\ &\approx \sqrt{b' - d + 4f_2}\left(1 + \frac{1}{b' - d + 4f_2}\frac{vb'^2}{2a_0^2} t\right)\,. \end{aligned}$$

Again, for clarity, let's denote in the next steps $s_1 = \sqrt{b' - d}$ and $s_2 = \sqrt{b' - d + 4f_2}$. Substituting into the formula for x and approximating, we get

$$\begin{aligned} x &= \frac{1}{2}\left[-b' - \frac{vb'^2}{a_0^2} t + d \pm s_1 s_2 \left(1 + \frac{vb'^2}{2s_1^2 a_0^2} t\right)\left(1 + \frac{vb'^2}{2s_2^2 a_0^2} t\right)\right] \\ &\approx \frac{1}{2}\left\{-s_1^2 - \frac{vb'^2}{a_0^2} t \pm s_1 s_2 \left[1 + \frac{vb'^2}{2a_0^2}\left(\frac{1}{s_1^2} + \frac{1}{s_2^2}\right) t\right]\right\}\,. \end{aligned}$$

Finally, we can express

$$x - x_0 = \frac{1}{2}\frac{vb'^2}{a_0^2}\left(\frac{s_1^2 + s_2^2}{2s_1 s_2} - 1\right) t\,.$$

The velocity of the lens is

$$u = \frac{x - x_0}{t} = \frac{1}{2}\frac{vf_1^2}{(a_0 - f_1)^2}\left(\frac{\frac{a_0 f_1}{a_0 - f_1} - d + 2f_2}{\sqrt{\frac{a_0 f_1}{a_0 - f_1} - d}\sqrt{\frac{a_0 f_1}{a_0 - f_1} - d + 4f_2}} - 1\right) \doteq -0.23\,\mathrm{cm\cdot s^{-1}}\,.$$

Note that in this fraction, the numerator contains the arithmetic mean, and the denominator contains the geometric mean of the values $\frac{a_0 f_1}{a_0 - f_1} - d$ and $\frac{a_0 f_1}{a_0 - f_1} - d + 4f_2$.

We still need to check in which direction the second lens moves, which determines the sign of the result. We defined its position x as its distance from the screen and the velocity u is then positive when moving away from the screen. Negative velocity u means that the second lens moves toward the screen. The ant is also moving toward the screen. The lens is moving in the same direction as the ant, so the result is $0.23\,\mathrm{cm\cdot s^{-1}}$.

Jaroslav Herman
jardah@fykos.org

Problem X.1 ... we are cleaning up the garden — 4 points

Autumn has come, and the leaves in our garden have started to fall. After raking them, we need to take them to a nearby pile, so we scoop them on a garden wheelbarrow and head towards the pile at a constant speed $v = 1.2\,\mathrm{m\cdot s^{-1}}$. We are pushing the wheelbarrow in front of us on a pavement made of interlocking paving stones, so there is always a small hole between the 15 cm tiles, where the speed of the wheelbarrow is reduced by 10 %. Determine the constant horizontal component of the force with which we need to push the wheelbarrow forward to make its average speed also v. The wheelbarrow together with the leaves has a mass of 19 kg.

Many leaves have fallen on Jarda's greenhouse.

The average speed of the wheelbarrow is the same as our walking speed, i.e. v. These speeds are constant, but the wheelbarrow periodically decreases its momentum. We have to compensate for this by applying a force to keep it at the average speed v.

We find the force F using the well-known relation

$$F = \frac{\Delta p}{\Delta t}\,,$$

where Δp is the change in momentum over time Δt. This time period corresponds to $\Delta t = \frac{l}{v}$, where l is the size of one tile.

On each tile the wheelbarrow loses momentum $\Delta p = m\Delta v$, where m is its mass and Δv is the change in speed. Let $v_{\max}$ denote the maximum speed of the wheelbarrow, and $v_{\min}$ its minimum speed. Then $\Delta v = v_{\max} - v_{\min}$ holds. At the same time the speed of the wheelbarrow is changing linearly in time (when it is not losing it on the edges of the tiles, so the average speed v has to be equal to the average

$$v = \frac{v_{\max} + v_{\min}}{2}\,.$$

We rewrite the last important piece of information in the problem statement as $v_{\min} = (1 - \eta)\,v_{\max} = 0.9\,v_{\max}$, where $\eta = 0.1$ is the ten-percent loss of the speed.

Then $v_{\max} = \frac{2v}{2-\eta}$ and $\Delta v = \eta v_{\max}$. By substituting into the original equation for the force, we get

$$F = \frac{mv\Delta v}{l} = \frac{2mv^2\eta}{l\,(2-\eta)} = 19.2\,\mathrm{N}\,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem X.2 ... watering the garden — 4 points

Jarda planted six fruit bushes next to a garden pond with a surface area of $A = 8\,\mathrm{m}^2$. These stand in a row, the first being $2.3\,\mathrm{m}$ from the pond and every other $1.0\,\mathrm{m}$ further than the previous one. Jarda always waters them using his ten-liter watering can that he refills at the pond. The total time it takes him to refill and pour one jug is $45\,\mathrm{s}$. Jarda walks at a speed of $v = 1.2\,\mathrm{m{\cdot}s^{-1}}$. In the meantime, the garden hose delivers water to the pond with a flow rate of $Q = 0.9\,\mathrm{l{\cdot}s^{-1}}$. After watering the last plant, he went to see how much the level of the pond had risen since he had started filling the first watering can. How much did he measure? If the level has dropped, submit a negative value. *Jarda found currants on sale.*

Let's determine the time it will take Jarda to water the garden. Let $N = 6$ be the number of bushes and $t_\mathrm{m} = 45\,\mathrm{s}$ be the time it takes to pour and refill the watering can, then the total time to handle the watering can is $t_\mathrm{k} = Nt_\mathrm{m}$.

The time it takes Jarda to walk back and forth is given by

$$t_\mathrm{ch} = 2\frac{1}{v}\left(\sum_{n=0}^{N-1} d_0 + nd\right) = \frac{2}{v}\left(Nd_0 + \frac{N\,(N-1)}{2}d\right)\,,$$

where $d_0 = 2.3\,\mathrm{m}$ is the distance of the first bush from the pond and $d = 1.0\,\mathrm{m}$ is the distance between the bushes.

The total watering time is

$$t = N\left(t_\mathrm{m} + 2\frac{d_0}{v} + d\frac{N-1}{v}\right)\,.$$

During this time, a volume of water Qt flows from the garden hose into the pond. However, Jarda has filled N gardening cans of volume $V_\mathrm{k} = 10\,\mathrm{l}$, so the increment in the volume of water in the pond is $\Delta V = Qt - NV_\mathrm{k}$.

The level has risen by

$$h = \frac{\Delta V}{A} = QN\frac{\left(t_\mathrm{m} + 2\frac{d_0}{v} + d\frac{N-1}{v} - \frac{V_\mathrm{k}}{Q}\right)}{A} = 2.83\,\mathrm{cm}\,.$$

Jaroslav Herman
`jardah@fykos.org`

Problem X.3 ... winding the cable 6 points

Jarda mows his garden with an electric mower. Once he has finished, he has to wind the entire cable of $L = 20\,\mathrm{m}$ onto the cable reel. It is wound onto a cylinder with a diameter $D = 40\,\mathrm{cm}$ whose axis of symmetry is horizontal. The cable attachment on the reel is close to the ground. The coefficient of friction between the ground and the cable is $f = 0.7$, and the length density of the cable is $\lambda = 220\,\mathrm{g{\cdot}m^{-1}}$. Determine the amount of work that Jarda will do when winding.

Jarda was laying the cable.

The frictional force acting on the cable is proportional to its length x, which remains in contact with the ground throughout the winding. Then

$$F = fmg = fgx\lambda \,,$$

where $m = \lambda x$ is the mass of the unwound cable and g is the gravitational acceleration.

The work to overcome the frictional force is found by using the integral as

$$W_\mathrm{F} = \int_0^L F\,\mathrm{d}x = fg\lambda \int_0^L x\,\mathrm{d}x = fg\lambda \left[\frac{x^2}{2}\right]_0^L = \frac{fg\lambda L^2}{2} \,.$$

Next, we raised the center of gravity of the entire cable by coiling. It is wound on a drum of diameter D, so the number of turns of the cable around it is $\frac{L}{\pi D} \doteq 15.9$. The center of gravity of the cable can thus be placed in the center of the reel, because we wrap a lot of turns, and the displacement of the center of gravity due to the incompletion of the last turn can be neglected. So we've done the work

$$W_\mathrm{g} = mgh = \lambda Lgh \,,$$

where $h = \frac{D}{2}$ is the height of the center of the reel (and therefore the center of gravity of the cable) above the ground. The total work done corresponds to

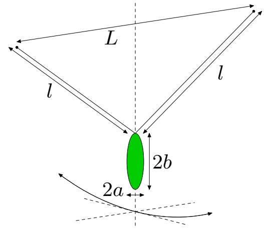
$$W = W_\mathrm{F} + W_\mathrm{g} = \lambda gL\left(\frac{D}{2} + \frac{fL}{2}\right) = 310\,\mathrm{J}\,.$$

Jaroslav Herman
jardah@fykos.org

Problem X.4 ... swinging cucumber 6 points

A cucumber in Jarda's greenhouse has the shape of an elongated ellipsoid with semi-axes $a = 3\,\mathrm{cm}$ and $b = 10\,\mathrm{cm}$ and hangs in the middle of a stem of length $2l = 90\,\mathrm{cm}$. The ends of the stem are distanced $L = 51\,\mathrm{cm}$ from each other and the weight of the cucumber is $m = 450\,\mathrm{g}$. Determine the period of the small oscillations around the equilibrium position when the deflection is perpendicular to the plane of the suspension.

Jarda likes cucumbers.



To find the period of the small oscillations we use the well-known relation for the physical pendulum

$$T = 2\pi\sqrt{\frac{J}{mgd}}\,,$$

where J is the moment of inertia of the oscillating body with respect to the axis of rotation, m is the mass of the body, g is the gravitational acceleration, and d is the distance of the center of gravity from the rotation axis.

The axis of rotation runs through the attachment points of the stems. The center of gravity of the cucumber is located at its center, so from the geometry of the problem we calculate the distance d as

$$d = b + \sqrt{l^2 - \frac{L^2}{4}} = 47\,\mathrm{cm}\,,$$

where $b = 10\,\mathrm{cm}$, $L = 51\,\mathrm{cm}$ and $l = 45\,\mathrm{cm}$ are the distances from the problem statement.

The moment of inertia of the cucumber with respect to this axis is found using Steiner's theorem as

$$J = J_T + md^2\,,$$

where J_T is the moment of inertia with respect to the axis passing through the object's center of gravity. For the ellipsoid in this particular geometry, we define it as $J_T = \frac{1}{5}m\left(a^2 + b^2\right)$.

After substituting into the original relation, we obtain

$$T = 2\pi\sqrt{\frac{J}{mgd}} = 2\pi\sqrt{\frac{\frac{a^2+b^2}{5} + \left(\sqrt{l^2 - \frac{L^2}{4}} + b\right)^2}{g\left(\sqrt{l^2 - \frac{L^2}{4}} + b\right)}} = 1.38\,\mathrm{s}\,.$$

Jaroslav Herman
jardah@fykos.org

FYKOS
UK, Matematicko-fyzikální fakulta
Ústav teoretické fyziky
V Holešovičkách 2
180 00 Praha 8

www: http://fykos.cz
e-mail: fykos@fykos.cz

FYKOS is also on Facebook
http://www.facebook.com/Fykos

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