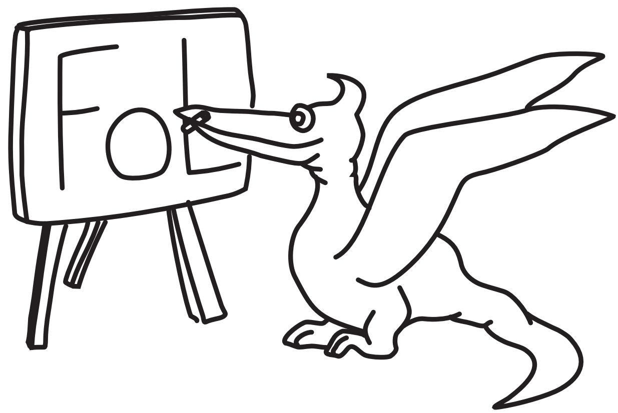
Solutions of 7^{th} Online Physics Brawl



Problem FoL.1 ... tough run

With what velocity should you run at the equator in order to weigh as much as possible (e.g. have the maximum possible weight, not mass) if you can choose the optimal direction?

Matěj's healthy method of gaining weight.

It's best to run against Earth's rotation, i.e. westwards, with the exact velocity that causes centrifugal force to be zero. The velocity should be equal and opposite to the velocity of rotating Earth at that point.

$$v = \omega R = \frac{2\pi R}{T} \doteq 464\,\mathrm{m{\cdot}s^{-1}}\,,$$

where $T = 24\,\mathrm{h}$ and $R = 6{,}380\,\mathrm{km}$.

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Problem FoL.2 ... electrobell

*Mirek's colleague bought a \$1 "small science" apparatus at a flea market. It's a small box with two wires. The ends of these wires are almost connected, leaving a small distance l between them. If we connect the wires to a powerful voltage source, we can create a short electric arc. The sound of this discharge is quite loud, that's why Mirek's colleague uses this device as a lunch bell. Find the largest possible distance (**in micrometres**) of the wires that allows the creation of an electric discharge for a source with peak voltage $325\,\mathrm{V}$. The dielectric strength of air is $D = 3\,\mathrm{MV{\cdot}m^{-1}}$.*

Mirek was looking forward to lunch.

The maximum voltage in the circuit is $U = 325\,\mathrm{V}$, so the largest distance between wires allowing the creation of a discharge is

$$l = \frac{U}{D} \doteq 110\,\mathrm{\mu m}\,.$$

Let us note that the presence of dust, shape of electrodes and other effect could increase this distance. For the purposes of the question, we omitted the Tesla coil inside the real device.

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Problem FoL.3 ... jump

The mass of the average human is $80\,\mathrm{kg}$. How many people would need to gather in one place and jump $1\,\mathrm{m}$ up at the same time in order to shift Earth's center by $0.1\,\mathrm{pm}$?

Matěj was jumping on a trampoline.

The common center of mass of Earth and the people doesn't move. At the moment when all people are $1\,\mathrm{m}$ above Earth's surface, the center of Earth should shift by $0.1\,\mathrm{pm}$.

$$N \cdot 1\,\mathrm{m} \cdot 80\,\mathrm{kg} \doteq 0.1\,\mathrm{pm} \cdot 5.97 \cdot 10^{24}\,\mathrm{kg}\,,$$

where we utilised the approximation $1\,\mathrm{m} - 0.1\,\mathrm{pm} \approx 1\,\mathrm{m}$.

$$N \doteq \frac{0.1\,\mathrm{pm} \cdot 5.97 \cdot 10^{24}\,\mathrm{kg}}{1\,\mathrm{m} \cdot 80\,\mathrm{kg}} \doteq 7.5 \cdot 10^{9}$$

That means we'd need the whole population of Earth.

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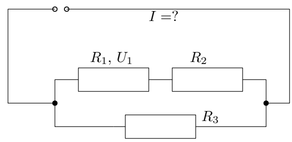
Problem FoL.4 ... there's a current flowing



What is the total current I flowing through the depicted circuit? Resistances in the diagram are $R_1 = 124\,\Omega$, $R_2 = 263\,\Omega$ and $R_3 = 454\,\Omega$. Voltage on the first resistor is $U_1 = 14.8\,\mathrm{V}$.

Enter the result in miliamperes.

Karel wanted you to review electrical circuits.

Let's denote the voltages and currents on each resistor as U_x and I_x. Ohm's law applied on the first resistor gives the current

$$I_1 = \frac{U_1}{R_1}\,.$$

Charge preservation (or Kirchhoff's first law) tells us that $I_1 = I_2$. This relation can be used to obtain the voltage on the second resistor

$$U_2 = R_2 I_2 = R_2 I_1 = \frac{R_2}{R_1} U_1\,.$$

According to Kirchhoff's second law, the total voltage on the upper branch is equal to the voltage on the lower branch. This leads to

$$U = U_3 = U_1 + U_2 = U_1 \left(1 + \frac{R_2}{R_1}\right)\,.$$

Now we can finally express the total current

$$I = I_1 + I_3 = \frac{U_1}{R_1} + \frac{U_1\left(1 + \frac{R_2}{R_1}\right)}{R_3} = U_1 \left(\frac{1}{R_1} + \frac{1}{R_3} + \frac{R_2}{R_1 R_3}\right)\,.$$

Plugging in the numbers gives $I \doteq 221\,\mathrm{mA}$.

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Problem FoL.5 ... a mirror problem

There's a $10\,\mathrm{m}$ wide, $5\,\mathrm{m}$ long rectangular room. In the middle of one of the longer walls, there's a mirror with width $1\,\mathrm{m}$ at eye level. Mikuláš is standing in front of the middle of the mirror at a distance $1\,\mathrm{m}$ from the mirror. What part of the room's area (in %) will he see in the mirror? Neglect the fact that part of this area won't be visible because he sees himself in front of it. Note: when computing the area, consider only the horizontal cross-section of the room at eye level. *Katka was looking out of a window during a lecture.*

It's clear that we're mainly interested in the two rays reflected from the mirror's border. Let's denote the distance of Mikuláš from the mirror by v and the half-width of the mirror by z. According to the law of reflection, we know that

$$\frac{v}{z} = \frac{y}{x}\,,$$

where y is the length of the room and x is the perpendicular distance between the point of incidence of the ray (at the mirror) and the ray's intersection with the wall behind Mikuláš. In this case, $x = 2.5\,\mathrm{m}$. It follows from the geometry of the problem that the ray will end up on the back wall of the room and separate the visible and invisible part of the room. After partitioning the visible part and summing up all areas, we find the area S of the visible part

$$S = 2zy + \frac{1}{2}2xy = zy\left(2 + \frac{y}{v}\right)\,,$$

numerically $S = 17.5\,\mathrm{m^2}$. The total area can be computed by simply multiplying its side lengths, numerically $S_\mathrm{p} = 50\,\mathrm{m^2}$. The ratio of visible to total area can be found simply as

$$p = \frac{S}{S_p}\,.$$

We can see that the result is $p = 35\,\%$.

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Problem FoL.6 ... supercooled water

There's a container with supercooled water at temperature $t = -8\,^\circ\mathrm{C}$. Initially there are no condensation nuclei, so it remains in the liquid state. Find the mass percentage of this liquid that will freeze after inserting a condensation nucleus. Neglect the heat capacity of the container. Use the specific heat capacity of water $c = 4{,}180\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$ and the latent heat of fusion $l = 334\,\mathrm{kJ{\cdot}kg^{-1}}$.

Remember: This is an easy problem, so keep it simple.

Karel attended a talk by doc. Bochníček on the properties of supercooled water.

Let's break this problem down from the viewpoint of heat balance. The supercooled water will start freezing the moment we insert condensation nucleus in the container and heat will be released during the fusion process. The whole volume of water (liquid or solid) will receive this heat and warm up to $0\,^\circ\mathrm{C}$. The heat balance can be written as $kml = mc\Delta t$, where m is the mass of water (cancels out), k is the fraction of water turned into ice, c is the specific heat capacity fo water and l is the latent heat of fusion of water. The left-hand side stands for the heat released by freezing water. The right-hand side stands for the heat received by the whole volume in order to heat up to $0\,^\circ\mathrm{C}$. Rearrangement of the heat balance equation leads to

$$k = \frac{c\Delta t}{l}\,.$$

For given values we obtain $k = 10\,\%$.

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Problem FoL.7 ... ecological

Imagine that electricity could only be produced by burning wood. Let's assume we have just enough wood to produce the amount of electricity necessary to power a tablet for one hour. How many sheets of paper could be made from this wood instead of burning it? The heating value of wood is $13\,\mathrm{MJ\cdot kg^{-1}}$. During production, distribution and storage of electricity, $80\,\%$ of the energy (heat) is lost. The tablet is fueled by a $3.6\,\mathrm{V}$ battery with current consumption of $0.5\,\mathrm{A}$. One sheet of paper weighs $5.0\,\mathrm{g}$ and double of this mass in wood is needed for its production.

Erik is saving the forests.

One sheet of paper of mass m requires $2m$ of wood for its production. Heating value of wood is H. Heat can be transformed to electricity with the efficiency of $\eta = 0.2$. Overall we obtain $2mH\eta$ of electrical energy from one sheet of paper.

The tablet consumes electrical energy UIt, where U is the voltage, I is the current and t is the time of one hour. The number of sheets of paper n necessary to keep the tablet running for one hour is

$$n = \frac{UIt}{2mH\eta} \doteq 0.25\,.$$

Thus we can say that a tablet would consume quarter of a sheet of paper per hour.

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Problem FoL.8 ... headshot

We fire a projectile with mass $m = 0.502\,\mathrm{g}$ from an air gun with mass $M_\mathrm{p} = 5\,\mathrm{kg}$. The projectile impacts a cuboid with mass $M = 182\,\mathrm{g}$, which slides and stops at distance $s = 4.8\,\mathrm{cm}$. The friction coefficient between the cuboid and surface is $f = 0.3$. Determine the muzzle velocity of the projectile.

Assume that the velocity of the projectile at the moment of impact is the same as the muzzle velocity and that the cuboid is far enough to be unaffected by gases emitted from the air gun. The cuboid lies on a horizontal surface.

Adapted by Karel from the article Fyzika jako zážitek ("Physics as an experience").

We can discard the mass of the gun, since we aren't computing the projectile's velocity based on recoil. However, the remaining parameters are relevant. First of all, we'll use the fact that we're dealing with a perfectly inelastic collision of the projectile and the cuboid. That means energy isn't conserved, but momentum is; both objects merge into one and start moving with a common velocity w after the collision. Let's denote the muzzle velocity of the projectile by v. Then we get

$$mv = (m + M)\,w \quad \Rightarrow \quad v = \frac{m+M}{m} w\,.$$

Of course, we don't know the initial velocity of the cuboid+projectile after the collision w. However, we know that due to friction, the cuboid will be moving with acceleration (deceleration) $a = fg$, where g is the acceleration due to gravity, until it stops. The formula for distance traversed during motion with constant acceleration is the well-known $s = at^2/2$; for velocity, it's $w = at$. Expressing the time t, we get

$$s = \frac{1}{2}\frac{w^2}{fg} \quad \Rightarrow \quad w = \sqrt{2fgs}\,.$$

All together, the muzzle velocity is given by

$$v = \frac{m+M}{m}\sqrt{2fgs} \doteq 193\,\mathrm{m\cdot s^{-1}}\,.$$

The muzzle velocity of our projectile is therefore $193\,\mathrm{m\cdot s^{-1}}$. We could neglect the fact that its mass increases the cuboid's mass in the collision – the result would be the same to three significant figures. We can't neglect the projectile's mass before the collision, though (that would give it zero momentum).

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Problem FoL.9 ... transportation acceleration

So one day Karel was casually riding the Prague metro and saw an information panel about trams. It said that the power of a tram car increased tenfold in the last decades. How many times did the maximum speed increase, assuming their weight didn't change and the resistive force are proportional to the second power of tram's velocity? In other words, we want to know the value of $k = v_1/v_0$, where v_0 is the original maximum speed and v_1 is the current one.

Karel saw a tram praising ad in the Prague metro.

The instantaneous power can be expressed as $P = Fv$, where F is the force exerted by the engine and v is the instantaneous velocity. When tram reaches its maximum speed, v can be viewed as a constant, therefore, the resistive force is also constant. Expression for the resistive force is $F = Cv^2$, where C is some constant. For a non-accelerated motion, the resulting force is zero. Thus the power can now be expressed as $P = Cv^3$. According to the information panel, the ratio of new and original power is $P_1/P_0 = 10$. Now we are ready to find the ratio of new and original maximum velocities

$$\frac{P_1}{P_0} = \frac{Cv_1^3}{Cv_0^3} \quad \Rightarrow \quad k = \frac{v_1}{v_0} = \sqrt[3]{\frac{P_1}{P_0}} = \sqrt[3]{10} \doteq 2.15\,.$$

Trams in Prague can achieve 2.15 times greater velocity.

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Problem FoL.10 ... closer than it seems

According to ground-based observatories on Earth, the parallax of the star Proxima Centauri is $p = 0.77\,\mathrm{arcsec}$. There's a planet orbiting this star at distance $r = 0.05\,\mathrm{au}$ (assume its orbit is approximately circular). Let's imagine there are intelligent beings living on this planet (let's call them Centaurs) and they measured the parallax of our Sun. The Centaurian definition of the parallax is, of course, based on their home planet's orbit. How large is the parallax they measured? Enter the result in ***arcseconds****.*

Mirek was pondering about extraterrestrial physical units.

The parallax of a star is (in a simplified way) defined as the following: let's construct a triangle between Earth, the Sun and the observed object. The parallax is the angle at the observed object–vertex. Since this angle will always be very small, we may use the small angle approximation (often called paraxial). If the orbital radius of the exoplanet corresponds to five hundredths of Earth's orbital radius, the parallax measured by the Centaurs will decrease proportionally, so it will be $p' = 0.77\,\mathrm{arcsec}/20 = 0.0385\,\mathrm{arcsec}$.

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Problem FoL.11 ... soda

How many times would the volume of a soda increase if all dissolved carbon dioxide suddenly turned to gas? A typical soda contains $8\,\mathrm{g\cdot dm^{-3}}$ of carbon dioxide. Consider the situation at $25\,^\circ\mathrm{C}$, $101.3\,\mathrm{kPa}$. *Štěpán spilled his drink on himself.*

The amount of carbon dioxide contained in the soda in moles is

$$n = \frac{m}{M}\,,$$

where m is the mass and M is the molar mass. In an ideal gas the amount of substance can be expressed as

$$n = \frac{V}{V_0}\,,$$

where V is the volume of the gas and V_0 is the molar volume at given conditions. The equation of state leeads us to $V_0 = RT/p = 24.5\,\mathrm{dm^3}$. Therefore the volume of gas is

$$V = \frac{V_0 m}{M}\,.$$

For one liter of soda we get $V = 4.45\,\mathrm{dm^3}$ of carbon dioxide gas. The total volume (liquid plus gas) is then $5.45\,\mathrm{dm^3}$, so the volume increased by the factor 5.45. The initial volume of dissolved carbon dioxide was neglected.

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Problem FoL.12 ... don't fall!

The two-wheeled vehicle segway maintains its vertical orientation by accelerating or decelerating. Neglect the mass of the vehicle and consider the driver a point with mass $65\,\mathrm{kg}$ at a distance $\frac{r}{2} = 1\,\mathrm{m}$ from the axis of rotation. What power does the motor need to provide when the segway needs to balance a tilt by $\alpha = 10^\circ$ to the front at speed $v = 10\,\mathrm{km\cdot h^{-1}}$? Use $g = 9.81\,\mathrm{m\cdot s^{-2}}$.
Michal was wondering about the new road signs.

Imagine the vehicle as a rod with length r that's tilted from the vertical by the angle α. The system is moving non-inertially (the vehicle makes turns, accelerates, decelerates...), which is why there's an inertial force F_1 acting on its center of mass (at distance $\frac{r}{2}$ from the axis of rotation). The force of gravity F_2 acting on the driver causes torque

$$M_2 = F_2 \frac{r}{2} \sin\alpha = mg\frac{r}{2}\sin\alpha\,.$$

Since the driver remains tilted by the same amount all the time, the net torque on the vehicle must be zero, so (with respect to the axis of rotation)

$$F_1 \cos\alpha \frac{r}{2} = mg\frac{r}{2}\sin\alpha\,.$$

Distance $r/2$ cancels out and we get

$$F_1 = mg\tan\alpha\,.$$

Now, from the point of view of the inertial system of the driver, we can identify the the inertial force as the force exerted by the motor. This force acts in the system the non-inertial system too, but it doesn't cause any torque because it acts at the axis of rotation. And when we know the force of the motor, the power can easily be computed using the well-known formula

$$P = F_1 v = mvg\tan\alpha\,.$$

Plugging in the numerical values, we get $P \doteq 312\,\mathrm{W}$.

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Problem FoL.13 ... shoot it down!

What's the probability (in percent) that if we shine a laser vertically upwards for a short time, the ray hits a passenger airplane? The average horizontal cross-section of an airplane is $S = 300\,\mathrm{m}^2$ and at each moment, there are approximately ten thousand airplanes in the air. Assume that airplanes are distributed homogeneously across the sky and fly at heights much smaller than the radius of Earth. *Matěj read about a terror attack.*

We may compute the probability as the ratio of total cross-section of all planes and Earth's surface area.

$$p = \frac{10{,}000 S}{4\pi R^2} = 5.9\cdot 10^{-9}\,.$$

The probability that the laser hits an airplane is $5.9\cdot 10^{-7}\,\%$.

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Problem FoL.14 ... Young's thermal stress

A copper rod of length $l = 12.3\,\mathrm{cm}$ and cross-sectional area $S = 1.02\,\mathrm{cm}^2$ is fixed so that its length cannot thermally increase. The linear coefficient of thermal expansion for copper is $\alpha = 1.70\cdot 10^{-5}\,\mathrm{K}^{-1}$ and Young's elastic modulus is $E = 117\,\mathrm{GPa}$. Find the force exerted by the rod on the holder due to thermal expansion. Give the result for $\Delta T = 15.0\,\mathrm{K}$ and only for one end of the rod. *Karel combined formulas.*

The idea behind this is problem is simple. The rod's length should increase, but it cannot, so it will deform. The deformation is due to force exerted on the rod by the holder and according to Newton's 3rd law, the rod exerts force of the same magnitude on the holder. This magnitude F can be expressed as $F = S\sigma$, where σ is the stress in the rod and S is the area of the contact surface, i. e., the cross-sectional area of the rod; radial expansion is negligible. The defining relation of Young modulus is $F = SE\varepsilon$, where ε is the strain. Assumption of linear dependence between length and temperature allows us to express

$$\varepsilon \approx \frac{\Delta l}{l} = \frac{(1+\alpha\Delta T)l - l}{l} = \alpha\Delta T\,,$$

where Δl is the absolute elongation. Combining all formulas we get

$$F = SE\alpha\Delta T\,.$$

The numerical result is $F = 3.04\,\mathrm{kN}$.

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Problem FoL.15 ... rubber bands

Consider two rubber bands (here, a rubber band is a single strip of rubber) with equal stiffness k and equal rest length. One of the bands breaks if acted upon by a force greater than F_1. The other band breaks if acted upon by a force greater than cF_1, where $c > 1$ is a constant. We take a weight and hang it from both rubber bands in parallel so that they don't break. Then, we slowly and continuously increase its weight (so that it does not oscillate) until the first band breaks. What's the minimal value of the constant c for which the second band doesn't break afterwards? *Michal was shooting rubber bands at people.*

According to the problem statement, the first rubber band breaks when there's a force F_1 acting on it. It follows that the weight must be acting with total force due to gravity $2F_1$ at the moment when the band breaks. At this point, the first band breaks and the weight is hanging only on the second band, which isn't in its equilibrium position.

We can view the resulting situation as a harmonic oscillator, which is initially in the position with maximum upwards displacement. The equilibrium position of this oscillator is a certain distance Δh below its current position. As is well known, the maximum downwards displacement of this weight must be located Δh below the equilibrium position. That's the position the weight will try to get to following the breaking of the first band; afterwards, it will oscillate between the two maximum positions.

Since the force acting upon the rubber band depends only on its displacement from the rest position, the force acting upon it will be maximum when the weight is in the position with maximum downwards displacement. In addition, we know that when the first band broke, the force acting upon the second band is F_1 and in the equilibrium position, the force acting upon the second band is $2F_1$. We can simply conclude that when the displacement is maximum downwards, the force is $3F_1$. Due to the previous reasoning, we know this is the maximum force which will act upon the second band during the oscillations.

The second rubber band has to be able to endure forces greater than $3F_1$. We can see that the constant c must satisfy $c > 3$.

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Problem FoL.16 ... bungee jumping

There's a bridge of height $h = 50\,\mathrm{m}$ and a bungee jumping rope with rest length $l = 10\,\mathrm{m}$ and stiffness $k = 80\,\mathrm{N{\cdot}m^{-1}}$. A man with mass m attaches himself with this rope to the bridge and jumps off (consider his initial velocity to be zero). What's the largest possible mass of this man such that the rope will stop him before hitting the surface below?

Michal is afraid to go bungee jumping.

First of all, let's derive a formula for the depth at which the rope stops the jumper. At this depth, the potential energy of the stretched rope will be equal to the potential energy of the jumper. Therefore, let's assume the rope stops the jumper at depth v (measured downwards from the top of the bridge). Then, we've got

$$\frac{1}{2}k(v - 10\,\mathrm{m})^2 = mgv\,,$$

where k is the stiffness of the rope, g the acceleration due to gravity and m is the mass of the jumper. The term $(v - 10\,\mathrm{m})$ at the left hand side of the equation has this form because the rest length of the rope is $10\,\mathrm{m}$.

We have two unknown quantities in this equation: the depth v, at which the jumper stops, and the mass of the jumper m. We can determine the first of these variables and determine the other one from this equation. Since we want to know the critical mass of the jumper, let's consider the case when the jumper is stopped by the rope exactly at ground level, that is, at depth $v = 50\,\mathrm{m}$. Now we can plug all the values to our equation and express the critical mass of the jumper as

$$m = \frac{\frac{1}{2}k(v - 10\,\mathrm{m})^2}{gv} \doteq 130.48\,\mathrm{kg}\,.$$

It's trivial that a jumper with larger mass won't be stopped by the rope in time.

For the rope to stop the jumper before hitting the ground, the mass of the jumper cannot be greater than $m = 130.48\,\mathrm{kg}$.

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Problem FoL.17 ... fidget spinner

A fidget spinner rotates with angular frequency $\omega = 12.34\,\mathrm{s^{-1}}$. Inner and outer radii of the ball bearing inside a fidget spinner are in ratio $k = 0.432$. What is the period of rotation of one ball inside the bearing around the center of the toy? The inner part of the bearing is static. Assume there is no slipping. *Matěj bought this autistic toy.*

We know the ratio $k = r/R$ of the inner radius r and the outer radius R. The instantaneous velocity of a point on the contact between a bearing ball and the outer part of the bearing is $v = \omega/R$. A point on the contact of the ball and the inner part of the bearing has zero instanteneous velocity, because the inner part is static. The velocity of the center of the ball is given by the average of these velocities, i. e. $v/2$. The distance between the center of the ball and the center of the bearing is $(r + R)/2$. The velocity of the center of the ball is therefore

$$\omega_k = \frac{v}{r + R} = \frac{\omega R}{r + R}\,.$$

Now we can express the period

$$T = \frac{2\pi}{\omega_k} = \frac{2\pi\,(r+R)}{\omega R} = \frac{2\pi}{\omega}\,(k+1)\ .$$

The numerical result is $T \doteq 0.729\,\mathrm{s}$.

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Problem FoL.18 ... insect in amber

Amber (with refractive index $n = 1.55$) is a transparent yellow-brown fossil resin. A small beetle, trapped and preserved within the stone, appears to be $2.25\,\mathrm{cm}$ below the surface when viewed directly from above. We are looking at the stone from a large distance. How far beneath the surface (in centimetres) is the beetle actually located?

Karel adapted a problem from Cutnell and Johnson: Physics 9e.

The beetle is at depth a below the surface of amber, but we see it as if the depth was b. Assume that our eyes are at the distance y above the surface. Think of a line perpendicular to the surface, going through the beetle. This line also goes exactly between our eyes since we are looking directly from above. We will denote the distance of each eye from the perpendicular line as x.

A light ray propagating from the beetle through the amber hits the surface at distance d, refracts and then goes straight to one of our eyes. Let's denote the angle of incidence and angle of refraction as α and β, respectively. Now we can write down a system of equations

$$\tan\alpha = \frac{d}{a}\,,$$
$$\tan\beta = \frac{x-d}{y} = \frac{d}{b}\,,$$

where the last expression was obtained by similarity of triangles, because we see the beetle in the direction of the refracted ray. By extending this ray we get a point of intersection with the perpendicular line and this point is b below the surface.

Snell's law leads to

$$\frac{\sin\alpha}{\sin\beta} = \frac{n_0}{n}\,,$$

where n_0 is the refractive index of air.

The accommodation range of eyes of an average adult is bounded from below by $\sim 15\,\mathrm{cm}$, therefore we can safely assume $x \ll y$. This allows us to use approximations $\tan\alpha \approx \sin\alpha \approx \alpha$. Thus we can plug the expressions for $\tan\alpha$ and $\tan\beta$ into Snell's law and obtain

$$\frac{\frac{d}{b}}{\frac{d}{a}} = \frac{n_0}{n}\,.$$

Rearranging for a gives

$$a = b\frac{n}{n_0}\,,$$

and for the given values we get $a \doteq 3.49\,\mathrm{cm}$.

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Problem FoL.19 ... swung rod

There's a rigid rod with length $l = 1.2\,\mathrm{m}$ and zero mass, which is attached at one end in such a way that it can rotate around this fixed end. There are three small spheres with equal mass attached to this rod (you may consider them point masses). One of the spheres is at the free end of the rod and the other two at 1/3 and 2/3 of its length. The rod is held horizontally at first. Then, we release it. What will be the velocity of its free end when it passes through the equilibrium position (the bottommost point)? The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Karel was teaching mechanics.

The moment of inertia of a point mass m with respect to an axis at distance d is $J = md^2$. If we denote the masses of the spheres as m, the moment of inertia of the whole system with respect to the fixed end of the rod is

$$J = m\left(\frac{l}{3}\right)^2 + m\left(\frac{2}{3}\right)^2 + ml^2 = \frac{14}{9}ml^2\,.$$

When the rod moves from the horizontal to vertical position, the potential energy that's released is

$$E = mg\frac{l}{3} + mg\frac{2}{3} + mgl = 2mgl\,.$$

It follows from the law of energy conservation that all this energy is converted to kinetic energy of the rod, which satisfies

$$E_\mathrm{k} = \frac{1}{2}J\omega^2 = \frac{1}{2}J\frac{v^2}{l^2}\,.$$

Substituting for E_k and J, we obtain the equation

$$2mgl = \frac{1}{2}\frac{14}{9}ml^2\frac{v^2}{l^2}\,,$$

from which we can easily express

$$v = \sqrt{\frac{18}{7}gl} \doteq 5.50\,\mathrm{m{\cdot}s^{-1}}\,.$$

The end of the rod will move with velocity $5.50\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.20 ... deviated

There's a pendulum consisting of a thin, rigid rod and a heavy weight attached to the end of the rod. The rod is then attached (by its other end) to another, horizontal rod. So, this second rod represents the axis of rotation of the pendulum. Now we rotate the second rod so that it is inclined at angle $\varphi = 30^\circ$ with respect to the horizontal plane (and it still represents the axis of rotation of the pendulum). Find the period of small oscillations T' of the pendulum and compare it with the period T of a pendulum with a horizontal axis of rotation. Give the ratio T'/T as the result.

Mirek was watching the new series Genius – Einstein.

Let's denote the gravitational acceleration by g. The movement of the deviated pendulum is constrained to a plane inclined at angle φ w.r.t. the horizontal plane. Projection of the gravitational acceleration on this plane is $g_{||} = g\cos\varphi$. The perpendicular component is $g_{\perp} = g\sin\varphi$ and is balanced out by forces inside the pivot (we assume zero friction, of course).

The period of small oscillations of a pendulum is related to g by

$$T \sim g^{-1/2}\,,$$

and for the deviated pendulum

$$T' \sim g_{||}^{-1/2}\,.$$

The ratio of these periods is

$$\frac{T'}{T} = \left(\frac{g\cos\varphi}{g}\right)^{-1/2} = \sqrt{\frac{1}{\cos\varphi}}\,.$$

Plugging in the numbers we get 1.075.

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Problem FoL.21 ... slinger

We've got a slingshot made by attaching two ends of a massless rubber band to two points $l_0 = 15\,\mathrm{cm}$ apart. Rest length of the rubber band is equal to this distance. We place a pebble with mass $m = 5\,\mathrm{g}$ in the middle of the rubber band and stretch it horizontally, forming the legs of an isosceles triangle with height $h_0 = 8\,\mathrm{cm}$. Then, we release the pebble. What's the maximum velocity the pebble will reach? The rubber band (as a whole) has stiffness $k = 50\,\mathrm{kg{\cdot}s^{-2}}$.

Mirek was remembering his childhood toy.

Before the rubber band is stretched, the potential energy $E_{\mathrm{p}0}$ is zero (even if the band was stretched between the attachment points, we could still take it to be zero). After stretching, its length increases from l_0 to

$$l = 2\sqrt{h_0^2 + (l_0/2)^2}\,.$$

The potential energy after stretching is

$$E_p = \frac{1}{2}k(l - l_0)^2 = \frac{1}{2}k(2\sqrt{h_0^2 + (l_0/2)^2} - l_0)^2$$

and when the pebble is launched, that's fully converted to kinetic energy

$$E_k = \frac{1}{2}mv^2 = E_p\,.$$

The pebble's launch velocity can now be expressed as

$$v = \sqrt{\frac{k}{m}}(2\sqrt{h_0^2 + (l_0/2)^2} - l_0)\,;$$

after plugging in the numbers, $v \doteq 6.9\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.22 ... energies of rotation

There are two identical homogeneous cylinders rotating with the same angular velocity ω. One cylinder, let's denote it A, is rotating around its main axis. The other cylinder B is rotating around a parallel axis with distance $4R/5$ from the center of the cylinder, where R is the radius of the cylinder. What's the ratio of rotational kinetic energies of the cylinders? We're interested in E_B/E_A, where E_A and E_B are rotational kinetic energies of cylinders A and B, respectively.

Karel and Lukáš were riding on carousel.

The rotational kinetic energy of a rigid body with moment of inertia J rotating with angular velocity ω is

$$E_\mathrm{k} = \frac{1}{2} J\omega^2 \,.$$

The moment of inertia of a cylinder with respect to its main axis (axis of symmetry) is

$$J = \frac{1}{2} mR^2 \,.$$

The kinetic energy of cylinder A can be computed easily as

$$E_A = \frac{1}{4} mR^2\omega^2 \,.$$

For the moment of inertia with respect to the axis displaced by d, we can utilise Steiner's (parallel axis) theorem $J = J_0 + md^2$. Substituting in the formula for kinetic energy, we get

$$E_B = \frac{1}{2}\left(\frac{1}{2}mR^2 + m\left(\frac{4}{5}R\right)^2\right)\omega^2 = \frac{57}{100} mR^2\omega^2 \,.$$

Therefore, the result is

$$\frac{E_B}{E_A} = \frac{\frac{57}{100}mR^2\omega^2}{\frac{1}{4}mR^2\omega^2} = \frac{57}{25} = 2.28 \,.$$

Ratio of rotational kinetic energies of the cylinders is 2.28.

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Problem FoL.23 ... the Ripper

Consider a barbell hanging horizontally in the air. For the purposes of this problem, we can imagine the barbell as a long thin rod with a small disk on each end. Let's charge one disk with charge $Q = 100\,\mu\mathrm{C}$, the other disk with charge $-Q$; the rod is perfectly insulating and no charge can be transfered to the rod. Then, let's activate a homogeneous electric field with intensity $E = 1\,\mathrm{MV{\cdot}m^{-1}}$ parallel to the rod in the direction from negative to positive charge. Determine the stress (in units Pa, with positive sign) in the middle of the rod caused by electrostatic forces. The diameter of the rod is $d = 2\,\mathrm{cm}$, the length of the rod $l = 1\,\mathrm{m}$. Neglect polarisation of dielectrics.

Can't get ripped by lifting weights? Rip the weights!

Let's utilise the superposition of electric fields. The charges, which can be treated as point charges due to small sizes of the disks compared to the whole barbell, attract each other with force

$$F_1 = \frac{kQ^2}{l^2} \,.$$

The rod is therefore compressed with force F_1, which corresponds to pressure

$$p_1 = \frac{F_1}{\frac{\pi d^2}{4}} = \frac{4kQ^2}{\pi d^2 l^2}\,.$$

At the same time, there's an external electric field acting on the charges. This field repels the disks from each other with force

$$F_2 = EQ\,.$$

The stress caused by this is

$$p_2 = \frac{F_2}{\frac{\pi d^2}{4}} = \frac{4EQ}{\pi d^2}\,.$$

Subtracting the stresses, we get

$$p_2 - p_1 = \frac{4Q}{\pi d^2}\left(E - \frac{k\ Q}{l^2}\right) \doteq 32{,}000\,\mathrm{Pa}\,,$$

which is the total stress in the rod. We can see that the rod wouldn't be torn apart, not even if it was made of soft plastic. Increasing the intensity of the electric field would just result in a spark discharge between the disks.

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Problem FoL.24 ... inhomogeneous

Consider a method of launching rockets by imparting all the necessary momentum at the moment of launch. A rocket with mass $m = 10\,\mathrm{t}$ is launched directly upwards at its escape velocity. How high above the Earth's surface will the rocket be when its velocity drops to half of the initial velocity? Express this result as a multiple of Earth's radius R (i.e. in units R). Neglect the effects of Earth's atmosphere and rotation.

Kuba meditating on the physics of balistic missiles.

The escape velocity is just enough for the rocket to stop at infinity. The law of energy conservation holds during the whole flight, so we can take the energy of the rocket at distance r from the center of Earth to be equal to the energy at infinity. That gives

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = 0\,,$$

where G is the gravitational constant and M is the mass of Earth. Therefore, we get

$$\frac{1}{2}mv^2 = \frac{GMm}{r}\,.$$

Taking the ratio of two such equations, we can see that

$$\left(\frac{v}{v_0}\right)^2 = \frac{R}{r}\,,$$

where $R = 6{,}378\,\mathrm{km}$ is the Earth's radius and v_0 is the initial, escape velocity. Now we can easily express the rocket's height at $v = v_0/2$ as

$$h = r - R = R\left(\frac{v_0}{v}\right)^2 - R = 3R\,.$$

The rocket's velocity drops to half at height $3R$.

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Problem FoL.25 ... Jáchym-style

Consider an isotropic point source of β radiation with activity $A = 3.4567\,\mathrm{MBq}$, located in a pool filled with a substance which dampens the radiation according to Lambert-Beer law $N(r) = N_0 \exp(-\mu r)$, where the absorption coefficient is $\mu = 1.1198 \cdot 10^{-1}\,\mathrm{m}^{-1}$. To what distance r from the source should we place a Geiger-Müller counter with detection window area $S = 2.7183 \cdot 10^{-6}\,\mathrm{m}^2$, if we want it to detect $N = 10$ particles per second on average? Assume that the detector registers all particles which hit the detection window.

Lukáš didn't want to use a computer just to download problems.

Let's at first neglect the substance dampening the radiation. The source is radiating isotropically into a spherical shell with surface area $4\pi r^2$, but we're only detecting the part of that radiation incident on a small surface with area S. The number of detected particles is

$$N = \frac{AS}{4\pi r^2}\,.$$

However, the radiation is dampened, so we need to multiply this formula by a Lambert factor. The resulting formula is

$$N = \frac{AS}{4\pi r^2}\exp\left(-\mu r\right)\,.$$

We've got a non-linear equation for r, which cannot be solved analytically. However, we can solve it either numerically or graphically. The graphical solution consists just of drawing a graph of the function $N(r)$ and reading out the value of r for the given N.

For a numerical solution, we can use a plethora of methods; let's use the interval halving method (bisection method). First, we choose a sufficiently large R so that the root of the equation

$$\frac{AS}{4\pi r^2}\exp\left(-\mu r\right) - N = f(r) = 0$$

which we're looking for would lie in the interval $(0, R)$. We can see that it's true e.g. if $f(0)$ and $f(R)$ have different signs. Then, we take the two halves of the interval. From these two halves, we pick one the root should definitely lie in and apply the same procedure again to that smaller interval until reaching the required precision.

The result is that the source should be at distance $r \doteq 0.269{,}35\,\mathrm{m}$.

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Problem FoL.26 ... drop the gallium into hot water

We've got $m_{\mathrm{Ga}} = 24\,\mathrm{g}$ of gallium and we'd like to perform an interesting experiment with it. We decide to prepare hot water with temperature $t_0 = 93\,^{\circ}\mathrm{C}$ and volume $V = 250\,\mathrm{ml}$. We place the gallium into an imperfect calorimeter with heat capacity $C = 95\,\mathrm{J{\cdot}K^{-1}}$ and then pour the hot water on it. What will the temperature (in degrees Celsius) of the calorimeter, gallium and water be after reaching thermodynamic equilibrium? Consider the calorimeter with water and *gallium an isolated, closed system. The initial temperatures of the gallium and the calorimeter were $t_1 = 22\,^\circ\mathrm{C}$. The latent heat of fusion of gallium is $l = 5.59\,\mathrm{kJ\cdot kg^{-1}}$, specific heat capacity as a solid $c_1 = 370\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$ and as a liquid $c_2 = 400\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$. The specific heat capacity of water is $c = 4{,}180\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$.*

Karel was thinking about the price of gallium, so he at least set this problem.

The problems seems to be clear at first. We're balancing the heat absorbed and given out. The only complication is caused by the fact that gallium melts at $t_t = 29.8\,^\circ\mathrm{C}$. Therefore, let's first determine by how much (let's denote it by ΔT) the water cools down when it heats up the gallium by $\Delta t_s = t_t - t_1$ and melts it. We can write the heat balance for this as

$$cV\varrho\Delta T = (C + c_1 m_{\mathrm{Ga}})\Delta t_s + m_{\mathrm{Ga}} l\,,$$

where ϱ is water density, which we'll consider to be $1\,\mathrm{g\cdot cm^{-3}}$. We find out that melting the gallium at the given initial temperature only cools down the water by very little. That means we'll have thermodynamic equilibrium between the calorimeter, water and gallium at a temperature higher than the melting point of gallium. We can now write the overall heat balance, with t denoting the final temperature.

$$(C + c_1 m_{\mathrm{Ga}})\Delta t_s + m_{\mathrm{Ga}} l + (C + c_2 m_{\mathrm{Ga}})(t - t_t) = (t_0 - t)cV\varrho$$

If we substitute Δt_s and express t, we obtain the equation

$$t = \frac{t_0 cV\varrho + t_t(C + c_2 m_{\mathrm{Ga}}) - m_{\mathrm{Ga}} l - (C + c_1 m_{\mathrm{Ga}})(t_t - t_1)}{(C + c_2 m_{\mathrm{Ga}}) + cV\varrho}\,.$$

After plugging in the numerical values, we get $t \doteq 86.4\,^\circ\mathrm{C}$.

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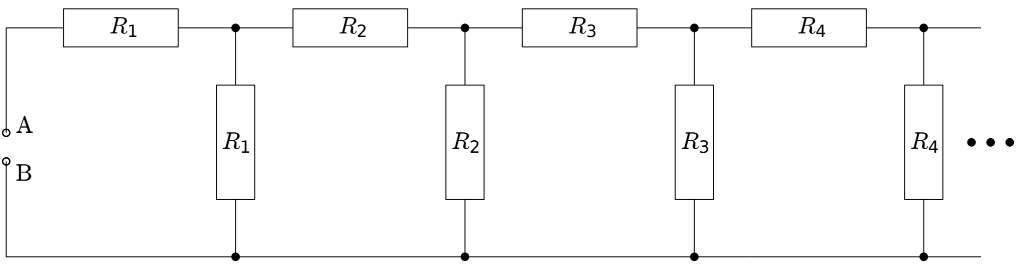
Problem FoL.27 ... another infinite circuit?



What's the resistance between points A and B of an infinite resistor network in the figure? The resistors' resistances are $R_i = 2^{i-1}R$ – each pair of resistors has the same resistance, which is twice as large as that of the previous pair. Compute the numerical result for $R = 3.002\,\Omega$.

Karel was thinking about variations of a standard problem.

We want to find the total resistance of the whole network; let's denote it by R_∞. One possibility is to compute partial resistances one by one and watch what number they converge to. That takes a lot of work, so it's much better to use a trick. Let's try to find the resistance of the whole network in some other way and get a quadratic equation, which can be solved to find R_∞.

In this case, let's imagine disconnecting the two resistors with resistance $R_1 = R$. How does the resulting circuit look? It's very similar to the previous circuit, but all resistors in it have doubled resistances. That's exactly what we need. Based on this idea, we can write the equation

$$R_\infty = R + \frac{2RR_\infty}{R + 2R_\infty}\,.$$

Now we only need to solve it

$$2R_\infty^2 - 3RR_\infty - R^2 = 0\,, \quad \Rightarrow \quad R_\infty = \frac{3 \pm \sqrt{17}}{4}R\,.$$

We need to think about which solution of the quadratic equation is the correct one. Since one solution is negative, we can easily pick the only positive solution. The total resistance is $R_\infty \doteq 5.346\,\Omega$.

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Problem FoL.28 ... medal

The organizers of FYKOS decided to award the top three participants with medals. The FYKOS medal is a flat cylinder consisting of three layers. The first layer is made of gold, the second layer is made of silver and the third one is copper. We know that the electrical resistance between the lower and upper base of the medal is the same as if all three layers were made of copper. We also know that the heat capacity of the medal is the same as if all three layers were made of gold. Find out the mass ratio of the FYKOS medal and another medal of identicial proportions that is made of silver only. Heat capacities, densities and resistivities of pure metals can be found on the Internet (or in printed engineering tables). *Jáchym thinks that a diploma is not enough.*

The height of the medal is $h = h_{\mathrm{Au}} + h_{\mathrm{Ag}} + h_{\mathrm{Cu}}$ where h_{Au}, h_{Ag} and h_{Cu} are the heights of the gold, silver and copper layers, respectively. We will make use of the relation for electrical resistance

$$R = \frac{l}{S}\zeta_{\mathrm{X}}\,,$$

denoting the resistivity as ζ_{X} to avoid confusing with mass density ϱ. The electrical resistance of the whole medal is given by the sum of respective resistances, which leads us to

$$\frac{h_{\mathrm{Au}}}{S}\zeta_{\mathrm{Au}} + \frac{h_{\mathrm{Ag}}}{S}\zeta_{\mathrm{Ag}} + \frac{h_{\mathrm{Cu}}}{S}\zeta_{\mathrm{Cu}} = \frac{h}{S}\zeta_{\mathrm{Cu}}\,,$$

where the RHS represents a copper medal of identical size. By substituting for h and multiplying by the cross-section S we get

$$h_{\mathrm{Au}}\zeta_{\mathrm{Au}} + h_{\mathrm{Ag}}\zeta_{\mathrm{Ag}} + h_{\mathrm{Cu}}\zeta_{\mathrm{Cu}} = h_{\mathrm{Au}}\zeta_{\mathrm{Cu}} + h_{\mathrm{Ag}}\zeta_{\mathrm{Cu}} + h_{\mathrm{Cu}}\zeta_{\mathrm{Cu}}\,,$$

$$h_{\mathrm{Au}} = h_{\mathrm{Ag}}\frac{\zeta_{\mathrm{Cu}} - \zeta_{\mathrm{Ag}}}{\zeta_{\mathrm{Au}} - \zeta_{\mathrm{Cu}}} = k_1 h_{\mathrm{Ag}}\,, \tag{1}$$

where we introduced the ratio k_1 of the heights of gold and silver layer.

The heat capacity of a body made of material X can be expressed as

$$C = mc_\mathrm{X} = V\varrho_\mathrm{X}c_\mathrm{X}\,.$$

Total heat capacity of the medal is again obtained as a sum of respective heat capacities

$$h_\mathrm{Au}S\varrho_\mathrm{Au}c_\mathrm{Au} + h_\mathrm{Ag}S\varrho_\mathrm{Ag}c_\mathrm{Ag} + h_\mathrm{Cu}S\varrho_\mathrm{Cu}c_\mathrm{Cu} = hS\varrho_\mathrm{Au}c_\mathrm{Au}\,.$$

By substituting for h and dividing by the cross-section S we get

$$h_\mathrm{Au}\varrho_\mathrm{Au}c_\mathrm{Au} + h_\mathrm{Ag}\varrho_\mathrm{Ag}c_\mathrm{Ag} + h_\mathrm{Cu}\varrho_\mathrm{Cu}c_\mathrm{Cu} = h_\mathrm{Au}\varrho_\mathrm{Au}c_\mathrm{Au} + h_\mathrm{Ag}\varrho_\mathrm{Au}c_\mathrm{Au} + h_\mathrm{Cu}\varrho_\mathrm{Au}c_\mathrm{Au}\,,$$

$$h_\mathrm{Cu} = h_\mathrm{Ag}\frac{\varrho_\mathrm{Au}c_\mathrm{Au} - \varrho_\mathrm{Ag}c_\mathrm{Ag}}{\varrho_\mathrm{Cu}c_\mathrm{Cu} - \varrho_\mathrm{Au}c_\mathrm{Au}} = k_2h_\mathrm{Ag}\,, \tag{2}$$

where we introduced the ratio k_2 of the heights of copper and silver layer.

The mass ratio of the FYKOS medal and a silver medal is

$$k = \frac{h_\mathrm{Au}S\varrho_\mathrm{Au} + h_\mathrm{Ag}S\varrho_\mathrm{Ag} + h_\mathrm{Cu}S\varrho_\mathrm{Cu}}{hS\varrho_\mathrm{Ag}} = \frac{h_\mathrm{Au}\varrho_\mathrm{Au} + h_\mathrm{Ag}\varrho_\mathrm{Ag} + h_\mathrm{Cu}\varrho_\mathrm{Cu}}{h_\mathrm{Au}\varrho_\mathrm{Ag} + h_\mathrm{Ag}\varrho_\mathrm{Ag} + h_\mathrm{Cu}\varrho_\mathrm{Ag}}\,.$$

Substituting for h_Au and h_Cu from equations (1) and (2) we establish the result

$$k = \frac{h_\mathrm{Ag}k_1\varrho_\mathrm{Au} + h_\mathrm{Ag}\varrho_\mathrm{Ag} + h_\mathrm{Ag}k_2\varrho_\mathrm{Cu}}{h_\mathrm{Ag}k_1\varrho_\mathrm{Ag} + h_\mathrm{Ag}\varrho_\mathrm{Ag} + h_\mathrm{Ag}k_2\varrho_\mathrm{Ag}} = \frac{1 + \frac{1}{\varrho_\mathrm{Ag}}\left(k_1\varrho_\mathrm{Au} + k_2\varrho_\mathrm{Cu}\right)}{1 + k_1 + k_2}\,.$$

For the given values we get $k \doteq 1.1$.

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Problem FoL.29 ... Dyson sphere under construction

A Dyson sphere is a hypothetical construction surrounding a star, built by an advanced civilisation in order to utilise all energy coming from their star. It should be a relatively thin shell with radius comparable to the distance of planets from the star. What would be the equilibrium temperature T_S of this shell compared to the equilibrium temperature T_P of a planet orbiting the same star on a circular orbit with distance equal to the radius of the Dyson sphere (in a system without the Dyson sphere)? Assume that the planet and the Dyson sphere are black bodies and all bodies in this problem radiate isotropically. Neglect the cosmic background radiation and the radiation of other space objects. As the result, compute the ratio $k = T_\mathrm{S}/T_\mathrm{P}$.

Karel was thinking about radiative heat transfer.

A Dyson sphere absorbs all solar radiation and radiates it in both directions (inwards and outwards). However, the sphere has to absorb everything it radiates inwards anyway (radiation which reaches the sun is negligible). In order to reach equilibrium, it has to radiate outwards the same power as the power radiated by the sun (let's denote it by P_S).

$$P_\mathrm{S} = 4\pi R^2\sigma T_\mathrm{S}^4\,,$$

$$T_\mathrm{S} = \sqrt[4]{\frac{P_\mathrm{S}}{4\pi R^2\sigma}}\,,$$

where R is the distance from the sun and $4\pi R^2$ is the surface area of the sphere. The sun radiates isotropically, so the radiation reaching the planet is

$$P = P_\mathrm{S}\frac{\pi r^2}{4\pi R^2}\,,$$

where πr^2 is the area of the planet's cross-section. The planet also has to radiate the same power from its surface

$$P = 4\pi r^2\sigma T_\mathrm{P}^4\,,$$

$$T_\mathrm{P} = \sqrt[4]{\frac{P}{4\pi r^2\sigma}} = \sqrt[4]{\frac{P_\mathrm{S}\frac{\pi r^2}{4\pi R^2}}{4\pi r^2\sigma}} = \sqrt[4]{\frac{P_\mathrm{S}}{16\pi R^2\sigma}}\,.$$

Now we can find the ratio

$$k = \frac{T_\mathrm{S}}{T_\mathrm{P}} = \sqrt[4]{4} = \sqrt{2}\,.$$

The equilibrium temperature of a Dyson sphere is $\sqrt{2}$ times the temperature of the planet.

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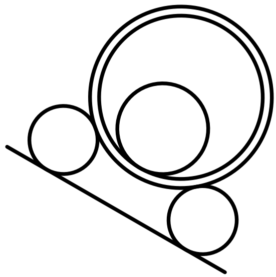
Problem FoL.30 ... too many cylinders

On an inclined plane with inclination angle 35°*, there's a system of three full cylinders and one hollow cylinder. Two smaller cylinders have radius* $r_1 = 0.1\,\mathrm{m}$ *and moment of inertia* $J_1 = 2\,\mathrm{kg{\cdot}m^2}$*, the middle cylinder has* $r_2 = 0.15\,\mathrm{m}$ *and* $J_2 = 10\,\mathrm{kg{\cdot}m^2}$*, and the hollow cylinder has* $r_3 = 0.3\,\mathrm{m}$*,* $d = 0.02\,\mathrm{m}$ *and* $J_3 = 20\,\mathrm{kg{\cdot}m^2}$*, where* r_3 *is the outer radius and* d *is the thickness of its walls. All cylinders are homogeneous. A rigid construction of negligible mass keeps the cylinders in the same relative positions, but allows them to rotate. Assume there is no slipping anywhere. What will be the distance travelled by this system during the initial* $15\,\mathrm{s}$ *after being released from rest?* *Because one cylinder is too mainstream.*



Let's denote the angular velocities of both small cylinders by ω_1. The whole system then moves with velocity $v = \omega_1 r_1$. The angular velocity of the hollow cylinder satisfies

$$\omega_3 = \omega_1\frac{r_1}{r_3}\,.$$

For the angular velocity of the middle cylinder, we get

$$\omega_2 = \omega_3\frac{r_3 - d}{r_2} = \omega_1\frac{r_1}{r_3}\frac{r_3 - d}{r_2}\,.$$

We can determine the masses of individual bodies from the formulae for moment of inertia

$$m_1 = \frac{2J_1}{r_1^2}\,,$$

$$m_2 = \frac{2J_2}{r_2^2}\,,$$

$$m_3 = \frac{2J_3}{\left(r_3^2 + (r_3 - d)^2\right)}\,.$$

The kinetic energy of one cylinder is $E_\mathrm{k} = \frac{1}{2}mv^2 + \frac{1}{2}J\omega^2$. If we substitute for the unknowns of each cylinder from the formulae above, we find that the total kinetic energy of the system satisfies

$$E_\mathrm{k} = \frac{3J_1}{r_1^2}v^2 + \frac{J_2}{r_2^2}v^2 + \frac{(r_3 - d)^2\, J_2}{2r_2^2 r_3^2}v^2 + \frac{J_3}{\left(r_3^2 + (r_3 - d)^2\right)}v^2 + \frac{J_3}{2r_3^2}v^2 = kv^2\,.$$

Initially, the system is at zero height with zero potential energy. When it travels distance x, it reaches height $h = -x\sin\alpha$, where α is the inclination angle of the plane. Its potential energy will be $E_\mathrm{p} = mgh = -mgx\sin\alpha$, where $m = 2m_1 + m_2 + m_3$. The total energy of the system is constant, so

$$E_\mathrm{k} + E_\mathrm{p} = kv^2 - mgx\sin\alpha = 0\,,$$

$$v^2 = \frac{mg\sin\alpha}{k}x\,.$$

The acceleration of the system is constant, so we can use the formulae $v = at$ and $x = \frac{1}{2}at^2$. Then, we have

$$a = \frac{mg\sin\alpha}{2k}\,.$$

Now we only need to substitute for the acceleration in the equation for x

$$x = \frac{mg\sin\alpha}{4k}t^2\,,$$

so $x \doteq 415\,\mathrm{m}$.

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Problem FoL.31 ... do I hear that?

They say Chuck Norris can go get his beer so fast he meets himself. However, how would that work if Chuck couldn't break the laws of physics? According to the theory of relativity, he can't move faster than light, so he can't see the photons he sent out (in vacuum, without reflection or refraction). However, could he hear himself? What's the minimum possible velocity with which he has to run if he wants to hear (faster, in reverse) what he said on the way there when running back? Chuck speaks with frequency $f = 200\,\mathrm{Hz}$ and the human ear (Chuck's ear too) can hear frequencies in the range $20\,\mathrm{Hz}$ through $20\,\mathrm{kHz}$. The speed of sound is $c = 340\,\mathrm{m{\cdot}s^{-1}}$.

Matěj imagined what it'd be like if Chuck Norris was subject to the laws of physics.

In order to hear himself, he has to move faster than the sound waves he's emitting. Therefore, Chuck's velocity v has to be higher than c. When he runs back with velocity v, he'll meet the sound waves in reverse order, so he'll hear what he said backwards. Due to the Doppler effect, the frequency of sound when the source and receiver move towards each other with velocities equal to v satisfies

$$\frac{f'}{f} = \frac{c+v}{c-v}\,.$$

The condition required for him to hear himself is

$$20\,\mathrm{Hz} \leq -f' \leq 20{,}000\,\mathrm{Hz}\,,$$

$$0.1 \leq -\frac{f'}{f} \leq 100\,.$$

It also follows from the Doppler law that when the source and receiver move towards each other with equal velocities, the absolute value of the incoming frequency can't be smaller than that of the original frequency. Therefore, we're only interested in the second condition, which says that the ratio of frequencies has to be smaller than 100. The negative sign of the frequency means Chuck hears himself in reverse. The resulting frequency has to be negative too. From the previous equation, we can express the velocity v as a function of the change in frequency

$$v = \frac{\frac{f'}{f} - 1}{\frac{f'}{f} + 1} c\,.$$

Substituting $f'/f = -100$, we obtain a condition for the velocity with which Chuck can move

$$v \geq \frac{101}{99} c = 346.9\,\mathrm{m\cdot s^{-1}}\,.$$

For any higher velocity, he'll hear himself clearly and the frequency he hears will approach the emitted (negative) frequency.

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Problem FoL.32 ... we'll be there in no time

*In a galaxy far far away, a gigantic spaceship that can move with velocity $v = 0.002c$ was built. After reaching this velocity, the spaceship set course for Earth. What will be the error made by an observer on Earth when estimating its distance, if he believes the spaceship to be a star and measures its redshift as $z = 0.005$? Assume the Hubble constant is $H = 70\,\mathrm{km\cdot s^{-1}\cdot MPc^{-1}}$. Enter the result in **megaparsecs**.* *Mirek was afraid of space invasion.*

The redshift and spaceship velocity are small enough, so we can use the linear approximation of Doppler effect

$$v_\mathrm{r} = zc\,,$$

where v_r is the velocity with which an object is moving away from us. According to Hubble's law,

$$v_\mathrm{r} = HD\,,$$

where D is the object's distance. According to an observer on Earth, the spaceship's distance is

$$D = \frac{zc}{H}\,,$$

but its real distance is

$$D' = \left(z + \frac{v}{c}\right)\frac{c}{H}\,.$$

The observer's error is

$$|D' - D| = \frac{v}{H} \doteq 8.6\,\mathrm{MPc}\,.$$

The spaceship will never reach us, of course – this holds for any object for which we observe redshift.

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Problem FoL.33 ... boring laboratory routine

We are sitting in a lab, measuring the spectral lines of hydrogen. One value of wavelength in the series of measurements is extremely low, $\lambda = 91.184\,\mathrm{nm}$. *Assuming the Bohr model of hydrogen is exact, what would be the distance of the electron whose transition caused this exceptional emission? We also assume that the electron was in a bound state, no matter how large the initial distance was. In your calculations, use the following constants: ionization energy of hydrogen* $E_0 = 13.598{,}4\,\mathrm{eV}$, *Planck constant* $h = 6.626{,}07\cdot 10^{-34}\,\mathrm{J{\cdot}s}$, *speed of light* $c = 2.997{,}92\cdot 10^{8}\,\mathrm{m{\cdot}s^{-1}}$ *and elementary charge* $e = 1.602{,}18\cdot 10^{-19}\,\mathrm{C}$.

Hint *Use all the qunatities to the full given precision!* *Mirek sees atoms.*

Using the Bohr model we can derive an expression for emitted wavelength

$$\frac{1}{\lambda} = \frac{E_0}{hc}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)\,,$$

where n_2 is an integer describing the initial energy level of the electron and n_1 is the final level. E_0 is the ionization energy (expressed in J). Basic knowledge of hydrogen series tells us that our observation fits into the Lyman series, i. e. $n_1 = 1$ (this can be confirmed by a short computation).

Electron-proton distance in the Bohr model is given by

$$r_n = r_0 n^2\,,$$

where $r_0 \doteq 52.9\,\mathrm{pm}$ is the so called Bohr radius. Expressing n_2 from the first equation (there's no point in rounding to the nearest integer due to very limited precision) and substituing to the formula for radius, we get

$$r = \frac{r_0}{1 - \frac{hc}{\lambda E_0}} \doteq 0.55\,\mu\mathrm{m}\,.$$

We can concluded that before emission, the radius of the hydrogen atom was comparable to the size of an average bacterium.

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Problem FoL.34 ... loud music

At a disco, there's loud music blaring from one loudspeaker with power 200 W*. Matěj doesn't like that music though, so he decides to play his own on his phone via its speaker with power* 4 W*. How many people will hear Matěj's track louder than the disco? Matěj is standing* 10 m *from the loudspeaker and the number density of people is* $2\,\mathrm{m^{-2}}$*.*

It's common to think about such things at a disco, right?

People will hear better the song whose intensity at that point is higher. Let's denote the power of the loudspeaker by P_D, the power of the phone speaker by P_M. We'll use the fact that intensity of sound I is actually power incident upon a unit area

$$I = \frac{P}{4\pi r^2}\,,$$

where r is the distance from a source with power P. Let's set up cartesian coordinates: the loudspeaker is located at the origin and the phone at the coordinates $(l, 0)$, where $l = 10\,\mathrm{m}$. The individual intensities depend on the position of the receiver,

$$I_\mathrm{D} = \frac{P_\mathrm{D}}{4\pi(x^2 + y^2)}\,,$$

$$I_\mathrm{M} = \frac{P_\mathrm{M}}{4\pi[(x - l)^2 + y^2]}\,.$$

Let's find the curve describing the border of the area where the phone has higher intensity

$$I_\mathrm{D} = I_\mathrm{M}\,,$$

$$\frac{P_\mathrm{D}}{4\pi\,(x^2 + y^2)} = \frac{P_\mathrm{M}}{4\pi\,((x - l)^2 + y^2)}\,,$$

$$P_\mathrm{D}(x - l)^2 + P_\mathrm{D}y^2 = P_\mathrm{M}x^2 + P_\mathrm{M}y^2\,,$$

$$x^2(P_\mathrm{D} - P_\mathrm{M}) + y^2(P_\mathrm{D} - P_\mathrm{M}) - 2xlP_\mathrm{D} + P_\mathrm{D}l^2 = 0\,,$$

$$x^2 - x\frac{2lP_\mathrm{D}}{P_\mathrm{D} - P_\mathrm{M}} + y^2 + \frac{P_\mathrm{D}l^2}{P_\mathrm{D} - P_\mathrm{M}} = 0\,,$$

$$\left(x - \frac{2lP_\mathrm{D}}{P_\mathrm{D} - P_\mathrm{M}}\right)^2 + y^2 = \frac{P_\mathrm{D}^2 l^2}{(P_\mathrm{D} - P_\mathrm{M})^2} - \frac{P_\mathrm{D}l^2}{P_\mathrm{D} - P_\mathrm{M}}\,,$$

$$\left(x - \frac{2lP_\mathrm{D}}{P_\mathrm{D} - P_\mathrm{M}}\right)^2 + y^2 = \frac{P_\mathrm{D}P_\mathrm{L}l^2}{(P_\mathrm{D} - P_\mathrm{M})^2}\,.$$

We managed to convert this equation to the form describing a circle with center away from the origin. This circle is known as the circle of Apollonius (it's defined by a fixed ratio of distances to 2 points) and its radius is

$$r^2 = \frac{P_\mathrm{D}P_\mathrm{L}l^2}{(P_\mathrm{D} - P_\mathrm{M})^2}\,.$$

The area of this circle is

$$S = \frac{\pi P_\mathrm{D}P_\mathrm{L}l^2}{(P_\mathrm{D} - P_\mathrm{M})^2} \doteq 6.54\,\mathrm{m^{-2}}\,.$$

The number of people hearing the phone with higher intensity is $S \cdot 2\,\mathrm{m^{-2}} \doteq 13$ people.

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Problem FoL.35 ... football player

What's the lowest velocity which a footballer has to impart to a ball located at the border of the penalty area in order to hit the crossbar? The distance to the goal is $d = 16.5\,\mathrm{m}$ and the crossbar is at height $h = 2.44\,\mathrm{m}$. Neglect air resistance and the dimensions of the ball and the crossbar.

Kuba wanted to cheat.

The minimum necessary velocity corresponds to minimum distance to the goal, so we should have the footballer stand straight in front of the goal and kick perpendicularly to the goal line. Now we're dealing with a 2D problem.

The ball follows a standard trajectory in a homogeneous gravity field, which is described by the equations

$$x = vt\cos\alpha\,,$$
$$y = vt\sin\alpha - \frac{1}{2}gt^2\,,$$

where v is the initial velocity of the ball and α is its elevation angle. The crossbar is hit at time t, so

$$d = vt\cos\alpha\,,$$
$$h = vt\sin\alpha - \frac{1}{2}gt^2\,.$$

This gives us two solutions (v, t), but only one of them has positive t. We get

$$v = \frac{d}{\cos\alpha}\sqrt{\frac{g}{2(d\tan\alpha - h)}} = d\sqrt{\frac{g}{d\sin(2\alpha) - h\cos(2\alpha) - h}}\,,$$

where we used the formulas for sine and cosine of a double angle.

Now, we've got velocity as a function of elevation angle only. The minimum of velocity occurs when the denominator under the square root is maximised, which means its derivative with respect to α must vanish. We get

$$\frac{\partial}{\partial\alpha}\left(d\sin(2\alpha) - h\cos(2\alpha) - h\right) = 2d\cos(2\alpha) + 2h\sin(2\alpha) = 0\,,$$
$$\tan(2\alpha) = -\frac{d}{h}\,.$$

We obtained a single extremum, which has to be the minimum, because $v(\alpha)$ is continuous and for angles 90° and $\arctan(h/d)$, we've got $v = +\infty$.

Since $\tan(2\alpha) < 0$ and $\alpha \in (0, 90^\circ)$, we also need $2\alpha \in (90^\circ, 180^\circ)$, when $\sin(2\alpha) > 0$ and $\cos(2\alpha) < 0$. Using the relations between goniometric functions, we can now express

$$\cos(2\alpha) = -\frac{1}{1 + \tan^2(2\alpha)} = -\frac{h}{\sqrt{d^2 + h^2}}\,,$$
$$\sin(2\alpha) = \sqrt{1 - \cos^2(2\alpha)} = \frac{d}{\sqrt{d^2 + h^2}}\,.$$

The last step is to substitute this back to the expression for velocity, which gives us the final expression for minimal velocity of the ball

$$v = \sqrt{g}\sqrt{\sqrt{d^2 + h^2} + h} \doteq 13.7\,\mathrm{m\cdot s^{-1}}\,.$$

Therefore $v \doteq 13.7\,\mathrm{m\cdot s^{-1}}$.

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Problem FoL.36 ... water over gold

Lord Waterboard wanted to have his waterproof castle filled with water. The drawbridge of the castle is rectangular (height $h = 3\,\mathrm{m}$ and width $s = 2\,\mathrm{m}$) and rotates around its bottom side. The drawbridge is drawn up and locked at the top; the lock will hold up against forces up to $F = 50\,\mathrm{kN}$ (the hinges at the bottom can withstand any force). What will the water level in the castle be at the moment when the lock breaks and the water flows out and onto the poor villagers? *Matěj was thirsty.*

Since the drawbridge acts as a lever, the lock is exposed to a force that differs from the outward force of the water pressure. For this reason, we need to calculate the force through its torque.

The maximum allowed torque at the drawbridge is$M = hF$. Let's denote the water level by H. The pressure at height x is $p(x) = (H - x)\varrho g$. There are two possible situations we can deal with depending on whether H is larger or smaller than h. In this specific case, $H > h$ (see the result below). The torque with which water pushes against the drawbridge can be computed by integrating

$$M = s\int_0^h xp(x)\,\mathrm{d}x = s\varrho g\left(\frac{1}{2}Hh^2 - \frac{1}{3}h^3\right)\,.$$

We obtain

$$hF = s\varrho g\left(\frac{1}{2}Hh^2 - \frac{1}{3}h^3\right)\,,$$

$$H = \frac{2s\varrho gh^3 + 6hF}{3s\varrho gh^2} \doteq 3.7\,\mathrm{m}\,,$$

which satisfies the condition $H > h$. If we tried to solve the other case (when the water level is below the top of the drawbridge), we'd get $H \doteq 3.6\,\mathrm{m}$, which contradicts the initial assumption.

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Problem FoL.37 ... unbent, unbowed, unsunken

On the flat water surface of a pond floats a hollow half-sphere with diameter $d = 30\,\mathrm{cm}$ and mass $m = 0.2\,\mathrm{kg}$. By a vertical impulse the sphere is forced into oscillation. What is the maximum kinetic energy of the impulse that won't submerge the half-sphere? Neglect the resistive forces of water. *Xellos placed a spoon on his tea.*

Oscillations are irrelevant, all we need to know is that the half-sphere will submerge when the surrounding water reaches its edge. This means that the maximum kinetic energy is equal to work done on the half-sphere that is necessary to move it from the equilibrium position to the critical position.

The resulting force exerted on the sphere is given as the difference between gravity and buoyancy

$$F(x) = \varrho\pi \left(Rx^2 - \frac{x^3}{3} \right) g - mg \,,$$

where x is the vertical size of the submerged part; we used the formula $V = \pi(Rx^2 - x^3/3)$ for the volume of a spherical cap. By integration we get

$$W = \varrho\pi(\frac{Rx^3}{3} - \frac{x^4}{12})g - mgx \,.$$

We need to integrate from equilibrium $F = 0$ to $x = R$. At the equilibrium, the balance equation is

$$Rx^2 - \frac{x^3}{3} = \frac{m}{\varrho\pi} \,,$$

which is a cubic equation with no obvious solution – numerically we get $x_0 \doteq 2.11\,\mathrm{cm}$. The maximum kinetic energy is then equal to $W(R) - W(x_0) \doteq 3.63\,\mathrm{J}$.

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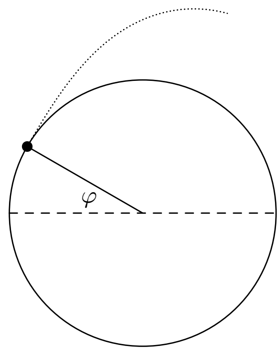
Problem FoL.38 ... in the rain

You're riding a bike, it's raining and there's a puddle in front of you. You don't have any mudguards, so you slow down to $3\,\mathrm{m\cdot s^{-1}}$*. What will be the maximum height (in* cm *above ground) to which the water will splash when flying off a tyre with radius* $35\,\mathrm{cm}$*? Neglect all resistive forces.* *Štěpán rode a bike in the rain.*



Due to centrifugal force, water flies off all points on the tyre. Consider a point at angle φ as shown in the figure. That is, water flies vertically upwards from the point at $\varphi = 0$ and horizontally from the point at $\varphi = \pi/2$, which is the highest point on the tyre.

The vertical component of velocity of water flying off from the point at angle φ is $v_y = v\cos\varphi$. The height of this point above the ground is $y_0 = r\sin\varphi + r$ – we shouldn't forget that the center of the wheel is at height r.

Using the well-known formula for motion under influence of gravity, we can obtain the maximum height reached from this point at angle φ

$$h(\varphi) = y_0 + \frac{v_y^2}{2g} = r\sin\varphi + r + \frac{v^2\cos^2\varphi}{2g} \,.$$

We're looking for the maximum height which the water can reach. The solution will clearly correspond to $\varphi \in \langle 0, \pi/2 \rangle$, since from any other point, water either flies off downwards or doesn't reach as high as from some other point.

The first derivative of h

$$\frac{\mathrm{d}h}{\mathrm{d}\varphi} = \cos\varphi \left(r - \frac{v^2}{g} \sin\varphi \right) .$$

An extremum can occur where the derivative is zero, that is, at points

$$\varphi_1 = \frac{\pi}{2} ,$$
$$\varphi_2 = \arcsin\left(\frac{gr}{v^2} \right) .$$

We can check that φ_1 is a minimum, while φ_2 is a maximum. The maximum height above the ground to which water splashes is

$$h(\varphi_2) = \frac{\left(gr + v^2\right)^2}{2gv^2} ,$$

so $h(\varphi_2) \doteq 87.5\,\mathrm{cm}$.

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Problem FoL.39 ... fire up the synapses

The magnetic field of Earth suddenly vanished and we need to replace it somehow. We know that the original field was approximately like that of a dipole and its magnitude at the equator is approximately given by the formula $|B| = B_0/R^3$, where $B_0 = 3.1 \cdot 10^{-5}\,\mathrm{T}$ and R is the distance from the center of Earth as a multiple of Earth's radius. We want to replace the field using a small coil (solenoid) inserted into the center of Earth. The coil has $N = 10^6$ loops and its radius is $\varrho = 1\,\mathrm{m}$. What current should flow through the coil if the new magnetic field is to be equal to the initial field? Assume that the permeability of all materials is the same as that of vacuum. *Mirek cannot count to four.*

The solenoid with current passing through it generates a magnetic field that's, at sufficient distances from the solenoid, the same as the field of a dipole. The dipole moment of one loop is given by

$$m = IS ,$$

where I is the current passing through it and S is the area enclosed by the loop. For our coil with N loops, the magnetic moment M can be expressed as

$$M = NI\pi\varrho^2 .$$

The field generated by a dipole moment $\boldsymbol{M}$ is described by the formula

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \left(\frac{3\boldsymbol{r}(\boldsymbol{M} \cdot \boldsymbol{r})}{|\boldsymbol{r}|^5} - \frac{\boldsymbol{M}}{|\boldsymbol{r}|^3} \right) ,$$

where μ_0 is the permeability of vacuum and $\boldsymbol{r}$ is the position vector pointing from the center of the dipole. The first term inside the brackets is zero on the equator ($\boldsymbol{M}$ is oriented north-south), so the magnitude of the coil's field at the equator is

$$|B| = \frac{\mu_0}{4\pi} \frac{M}{r^3} = \frac{\mu_0}{4\pi} \frac{NI\pi\varrho^2}{r^3} .$$

By comparing it with Earth's magnetic field, we get

$$\frac{B_0 R_\mathrm{E}^3}{r^3} = \frac{\mu_0}{4\pi} \frac{N I \pi \varrho^2}{r^3}\,,$$

and from this, the current can be expressed as

$$I = \frac{4 B_0 R_\mathrm{E}^3}{\mu_0 N \varrho^2}\,.$$

After plugging in all numbers, we get $I \doteq 2.6 \cdot 10^{16}\,\mathrm{A}$. We can safely claim that even if the coil didn't melt in Earth's core by some miracle, it would definitely melt because of the current flowing through it (melt is a serious understatement).

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Problem FoL.40 ... efficient ride

An electric car is driving on a level road. Its effective cross-section is $S = 2\,\mathrm{m}^2$ and its drag coefficient is $C = 0.2$. Its motor has efficiency $\eta = 40\,\%$. What's the optimal constant velocity (which maximises the car's range), if the power consumed inside the car (by air conditioning, radio,...) is a constant $P_0 = 400\,\mathrm{W}$? The density of air is $\varrho = 1.29\,\mathrm{kg{\cdot}m^{-3}}$.

Matěj dreams of driving a Tesla.

Using Newton's formula for air drag

$$F_\mathrm{o} = \frac{1}{2} C S \varrho v^2\,.$$

The power spent when driving with velocity v is

$$P_v = F_\mathrm{o} v = \frac{1}{2} C S \varrho v^3\,.$$

The total power consumption of the car is the sum of this power (divided by efficiency) and the power spent on appliances inside the car. That means

$$P = \frac{1}{\eta} P_v + P_0\,.$$

If the car's battery capacity is E, it can keep going for time $t = \frac{E}{P}$ and its range is

$$s = vt = \frac{Ev}{P} = \frac{Ev}{\frac{1}{2\eta} C S \varrho v^3 + P_0}\,.$$

The first derivative of the distance with respect to velocity vanishes at the maximum,

$$\frac{\mathrm{d}s}{\mathrm{d}v} = \frac{E \frac{1}{2\eta} C S \varrho v^3 + E P_0 - E v \frac{3}{2\eta} C S \varrho v^2}{\left(\frac{1}{2\eta} C S \varrho v^3 + P_0\right)^2} = 0\,,$$

$$\frac{1}{2\eta} C S \varrho v^3 + P_0 - \frac{3}{2\eta} C S \varrho v^3 = 0\,,$$

$$v = \sqrt[3]{\frac{P_0 \eta}{C S \varrho}}\,.$$

Numerically, $v \doteq 6.77\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.41 ... hot day

During the spring equinox a little boat is swimming in the equatorial seas. The boat has the shape of a rectangle with surface area $5\,\mathrm{m^2}$. The Sun shines on the surface of Earth with intensity $1.3\,\mathrm{kW{\cdot}m^{-2}}$. How much solar energy in MJ will the boat receive from sunrise to sunset? *Štěpán gazed on hot metal roofs.*

The first step is to realize that the angle of incidence of sun rays is changing throughout the day. Right after the sunrise and right before the sunset, the incident power will be minimal, while at noon it will reach the maximum value. Let us assume that the speed of the boat is negligible in comparison to Earth's rotation.

For an angle of incidence α the power can be expressed as $P(\alpha) = SI\sin(\alpha)$, where S is the surface area of our boat and I is the intensity of solar radiation. The angle α changes linearly with time and the dependence can be written as $\alpha(t) = \pi t/T$, where $T = 12\,\mathrm{h}$ is the length of one day. The total energy received by the boat is given by the integral

$$E = \int_0^T P(\alpha)\,\mathrm{d}t = SI\int_0^T \sin\left(\pi\frac{t}{T}\right)\,\mathrm{d}t = \frac{2SIT}{\pi}\,.$$

The boat will receive $178.8\,\mathrm{MJ}$ in the form of solar energy, which is equal to $64\,\%$ of energy received during twelve hours of perpendicular irradiation.

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Problem FoL.42 ... all the way up, please

What's the minimum length (in kilometres) of a space elevator built at the equator, so that it doesn't collapse under gravity? Consider the elevator to be just a few straight ropes leading to space. The elevator is homogeneous and doesn't contain any extra weight at the end.

We're not interested in the trivial solution of zero length. *Štěpán forgot how to use a lift.*

The minimum length corresponds to the case in which the whole rope is vertical.

In order to simplify the solution, let's define linear density of the elevator σ.

Now, let's focus on a small segment of the elevator with length $\mathrm{d}r$ and distance r from the center of Earth. The mass of this segment is $\mathrm{d}m = \sigma\mathrm{d}r$. This small segment is attracted to Earth by gravitational force $G\frac{M\mathrm{d}m}{r^2}$, where G is the gravitational constant and M is the mass of Earth. At the same time, this segment is repelled from Earth by centrifugal force $-\omega^2 r\mathrm{d}m$, where ω is the angular velocity of Earth's rotation. These forces act in opposite directions, so we need to change the signs of one of them.

Summing up these two (counteracting) forces, we get the total force acting on a small segment of the elevator

$$\mathrm{d}F = \sigma\left(G\frac{M}{r^2} - \omega^2 r\right)\mathrm{d}r\,.$$

At small distances, the gravitation is larger, so $\mathrm{d}F > 0$. At some height, the so-called geostationary orbit, $\mathrm{d}F = 0$. For all higher segments, the centrifugal force will be stronger and $\mathrm{d}F < 0$. We're interested in the length of the elevator, measured from the surface of Earth with radius R up to some height h above the surface, which leads to the net force summed over all segments being zero. That means the gravitational forces at small heights and centrifugal forces at large heights cancel out and the elevator will neither fall down nor be ripped off at the base. That means

$$\int_R^{R+h} \mathrm{d}F = 0\,,$$

$$\sigma\left(\frac{hGM}{R(R+h)} - \frac{1}{2}h\omega^2(2R+h)\right) = 0\,.$$

In this last equation, the only unknown is h. After some manipulation, we obtain a quadratic equation with a suitable solution

$$h = \sqrt{\frac{2GM}{R\omega^2} + \frac{R^2}{4}} - \frac{3R}{2}\,,$$

so $h \doteq 144{,}000\,\mathrm{km}$.

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Problem FoL.43 ... shower

What's the maximum useful running time for a shower fed from a cylindrical boiler with cross-section $S = 0.8\,\mathrm{m}^2$ and height $H = 1.5\,\mathrm{m}$? The showerhead is connected directly to the bottom of the boiler, the area through which water flows out of the showerhead is $s = 0.8\,\mathrm{cm}^2$. Consider the minimum volumetric flow rate for which showering is possible to be $Q_0 = 2\,\mathrm{dl{\cdot}s^{-1}}$. Assume that there's no water flowing into the boiler and all water flowing into the showerhead comes from the boiler.

Enter the result in minutes. *There wasn't enough water for Kuba!*

The flow of water must satisfy Bernoulli's equation

$$Q = Sv = su\,,$$

$$\frac{1}{2}\varrho v^2 + h\varrho g = \frac{1}{2}\varrho u^2\,,$$

where v is the speed with which the water level in the boiler drops, u is the speed with which water flows out of the showerhead, ϱ is the density of water, h the height of the water level in the boiler and g is the acceleration due to gravity. Since $S \gg s$, it follows from the continuity equation that $v \ll u$, so we may neglect the term with v in the Bernoulli equation.

From this, we can express v as

$$v = \frac{s}{S}\sqrt{2gh}\,.$$

Now, we can compute the minimum allowed height of water in the boiler

$$Q_0 = Sv_0 = s\sqrt{2gh_0}\,,$$

which gives us

$$h_0 = \frac{Q_0^2}{2gs^2} \doteq 32\,\mathrm{cm}\,.$$

Now we have to solve the differential equation

$$-\frac{\mathrm{d}h}{\mathrm{d}t} = v = \frac{s}{S}\sqrt{2gh}\,,$$

which can be done as

$$\frac{s}{S}\sqrt{2g}\int\limits_0^t \mathrm{d}t = -\int\limits_H^{h_0} \frac{\mathrm{d}h}{\sqrt{h}} = 2\left(\sqrt{H} - \sqrt{h_0}\right)\,,$$

$$t = \frac{S}{s}\sqrt{\frac{2}{g}}\left(\sqrt{H} - \sqrt{h_0}\right) = 50\,\mathrm{min}\,.$$

We can keep showering for 50 minutes using only water from the boiler. In reality, water from the boiler would be mixed with an approximately equal amount of tap water, which would give us an hour and half (provided a suitable technical execution).

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Problem FoL.44 ... space race

*Two spacehips, USS Annie and USS Bonnie, are racing. Both are approaching the finishing line with their maximum speeds $v_\mathrm{A} = c/4$ and $v_\mathrm{B} = c/2$. An observer at rest at the finishing line at time 0 observes Bonnie at distance $d_\mathrm{B} = 250\,\mathrm{km}$ and Annie at distance $d_\mathrm{A} = 100\,\mathrm{km}$. Determine the absolute difference (positive number) between the times when the spaceships reach the finishing line according to this observer in **milliseconds**.*

According to Mirek, placings in a race are relative.

At the moment when we see Annie at distance d_A, the spaceship is actually closer, because the light signal reached us with a delay. Let's denote the travel time of the signal by t and the real distance of Annie from the goal by d'_A. The signal travelled for

$$t = \frac{d_\mathrm{A}}{c}\,,$$

so the real distance is

$$d'_\mathrm{A} = d_\mathrm{A} - v_\mathrm{A}t = d_\mathrm{A}\left(1 - \frac{v_\mathrm{A}}{c}\right)\,.$$

We can similarly compute the real distance of Bonnie

$$d'_\mathrm{B} = d_\mathrm{B} - v_\mathrm{B}t = d_\mathrm{B}\left(1 - \frac{v_\mathrm{B}}{c}\right)\,.$$

The difference between finish times of the spaceships is

$$|t_\mathrm{A} - t_\mathrm{B}| = \left|\frac{d'_\mathrm{A}}{v_\mathrm{A}} - \frac{d'_\mathrm{B}}{v_\mathrm{B}}\right| = \left|\frac{d_\mathrm{A}}{v_\mathrm{A}}\left(1 - \frac{v_\mathrm{A}}{c}\right) - \frac{d_\mathrm{B}}{v_\mathrm{B}}\left(1 - \frac{v_\mathrm{B}}{c}\right)\right|\,.$$

Numerically, we get $|t_\mathrm{A} - t_\mathrm{B}| \doteq 0.167\,\mathrm{ms}$. The first ship to reach the goal is USS Bonnie.

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Problem FoL.45 ... gray plates

There are two parallel infinite planes in vacuum. The first plane has fixed temperature $T_1 = 200\,\mathrm{K}$ and reflectivity $R_1 = 1/2$ (the ratio of reflected to incident light intensity). The second plane has fixed temperature $T_2 = 300\,\mathrm{K}$ and reflectivity $R_2 = 1/3$. What will be the temperature of a third plane with reflectivity $R = 1/3$ placed in parallel between them?

Assume that all planes radiate as black bodies (have emissivity 1), they just partially reflect incident light. The transmissivities of all bodies are zero.

Kuba wanted to quantify the cooling effect of reflection

A radiated ray will keep being partially reflected between planes and partially absorbed. Therefore, part of the energy radiated by each plane returns back to it. We must compute what part of radiated heat is actually transferred to other planes.

When plane R radiates a ray with intensity I in the direction of plane R_1, the ray reflected from that plane has intensity $R_1 I$, which means that plane R_1 absorbed $(1-R_1)I$. The reflected ray is reflected a second time from plane R, which leads us to the original situation, just with intensity I replaced by $R_1 R I$. This way, we could keep going to get a geometric series for intensity absorbed at plane R_1 in the form

$$I_1 = \sum_{k=0}^{\infty} I(RR_1)^k(1 - R_1) = I\frac{1 - R_1}{1 - RR_1}\,.$$

The complement of this intensity $I - I_1$ is equal to the intensity absorbed by plane R.

Now we've got full information about energy transfer between planes. Replacing R_1 by R_2, we obtain the formula for intensity transferred from plane R to plane R_2 and replacing R_i by R, we obtain the formula for intensity transferred from plane R_i to plane R.

In stationary state, the total radiated energy is equal to total incident energy. This holds for radiated and incident power and therefore for intensities of exchanged radiation. After multiplying terms from Stefan-Boltzmann law by coefficients given by multiple reflection between planes, we may write the law of energy conservation in the form

$$\sigma T^4\left(\frac{1 - R_1}{1 - RR_1} + \frac{1 - R_2}{1 - RR_2}\right) = \sigma T_1^4\frac{1 - R}{1 - RR_1} + \sigma T_2^4\frac{1 - R}{1 - RR_2}\,.$$

After plugging in all reflectivities, the solution is

$$T = \sqrt[4]{\frac{16T_1^4 + 15T_2^4}{27}} \doteq 272\,\mathrm{K}\,.$$

This result says that if $T_1 = T_2$, the resulting T would be higher. That's caused by asymmetry between planes R and R_1, where both the energy radiated by plane R_1 and that radiated by plane R flow mostly towards plane R. It's all counter to physical intuition and says that the emissivity of a grey body is actually less than 1.

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Problem FoL.46 ... do not drown!

There's a river of width $d = 20\,\mathrm{m}$ with a parabolic velocity profile, meaning that at the distance y from either bank, the speed of the current is $v(y) = 4v_0 \frac{y}{d}\left(1 - \frac{y}{d}\right)$, where $v_0 = 1\,\mathrm{m{\cdot}s^{-1}}$ is the speed in the middle. The vertical velocity profile is constant. How long would it take us to swim across the river if we stay on a trajectory perpendicular to the bank? In the absence of any current our swimming speed would be $w = 2v_0$. *Kuba was carried away by the current.*

In order to stay on the given trajectory, the swimmer must compensate for the current speed. Let us denote the angle between the vector of the swimmer's velocity and the perpendicular trajectory as α. This angle appears in the formula for the x component of swimmer's velocity

$$w \sin\alpha = v(y) = 4v_0 \frac{y}{d}\left(1 - \frac{y}{d}\right) .$$

The y component can be obtained from the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = w\cos\alpha = w\sqrt{1 - \left[\frac{4v_0}{w}\frac{y}{d}\left(1 - \frac{y}{d}\right)\right]^2} .$$

Here we used $\cos\alpha = \sqrt{1 - \sin^2\alpha}$.

The solution of this differential equation will lead us to the expression for travel time τ. Plugging in the value $w = 2v_0$ and substituing $\psi = y/d$, we get

$$\tau = \frac{1}{w}\int\limits_0^d \frac{\mathrm{d}y}{\sqrt{1 - \left[\frac{4v_0}{w}\frac{y}{d}\left(1 - \frac{y}{d}\right)\right]^2}} = \frac{d}{w}\int\limits_0^1 \frac{\mathrm{d}\psi}{\sqrt{1 - 4\psi^2\left(1 - \psi\right)^2}} \doteq 1.078\frac{d}{w} \doteq 10.8\,\mathrm{s} .$$

The elliptical integral has to be solved numerically. We can now see that the resulting time is only by $0.8\,\mathrm{s}$ longer than it would be if the river was not flowing.

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Problem FoL.47 ... pressure cooking

A pressure cooker is a closed pot with walls of thickness $t = 5\,\mathrm{mm}$, thermal conductivity $\lambda = 9\,\mathrm{W{\cdot}m^{-1}{\cdot}K^{-1}}$, volume $V = 3\,\mathrm{l}$ and surface area $S = 4\,\mathrm{dm^2}$. There is a single hole in its walls with diameter $d = 4\,\mathrm{mm}$. We start heating the pot with power $P = 7\,\mathrm{kW}$, pour $V_\mathrm{v} = 1\,\mathrm{l}$ of water inside, close the lid and bring it boil. The room temperature is $T_\mathrm{i} = 20\,\mathrm{^\circ C}$. Neglect the dependence of boiling point on pressure. What will be the pressure increase (with respect to atmospheric pressure) inside the pot? *Xellos knows how to cook water.*

Let's assume that the system of pot and steam is in thermodynamic equilibrium, so the power P is spent only on heat loss through the walls of the pot and evaporation of water. If the temperature of the pot and water is equal to the boiling point $T = 100\,\mathrm{^\circ C}$, the power lost through the walls of the pot is

$$P_\mathrm{e} = \frac{\lambda S}{t}(T - T_\mathrm{i}) ,$$

so the water evaporates at a rate

$$\frac{\mathrm{d}m}{\mathrm{d}\tau} = \frac{1}{l}\left(P - \frac{\lambda S}{t}(T - T_\mathrm{i})\right) ,$$

where $l \doteq 2.26\,\mathrm{MJ{\cdot}kg^{-1}}$ is the latent heat of vaporisation of water. Since the volume of steam is much larger than volume of water with equal mass, we need an approximately equal mass of water to escape from the pot through the hole; this gives us the velocity of steam escaping through the hole

$$v = \frac{\mathrm{d}m}{\mathrm{d}\tau}\frac{1}{\varrho}\frac{4}{\pi d^2} .$$

We can tie this velocity to an increase in pressure based on Bernoulli equation

$$\Delta p = \frac{1}{2}\varrho v^2 .$$

The density of steam ϱ at temperature T is given by the ideal gas equation of state as

$$\varrho = \frac{Mp}{RT} ,$$

where $M = 18\,\mathrm{g{\cdot}mol^{-1}}$ is the molar mass of water and p is the pressure (we may assume the increase in pressure is sufficiently small, so p is approximately equal to atmospheric pressure). All together, we get

$$\Delta p = \frac{RT}{Mp}\frac{8}{\pi^2 d^4 l^2}\left(P - \frac{\lambda S}{t}(T - T_\mathrm{i})\right)^2 \doteq 1.6\,\mathrm{kPa} .$$

The assumption $T = 100\,^\circ\mathrm{C}$ isn't completely correct, since the boiling point depends on pressure. However, we can see that we really have $\Delta p \ll p$, so the real boiling point of water in the pot is close to the standard one and the result is approximately correct.

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Problem FoL.48 ... thirteen barrels

Consider a hermetically sealed container with known volume containing a gas with known pressure. We also have a barrel with volume equal to one thirteenth of the volume of the container. First, we connect the barrel to a source of gas (which fills it with gas with fixed pressure) and wait for the pressure in the barrel to equalise. Then, we disconnect the barrel from the source, connect it to the container and wait for the pressures to equalise. Finally, we disconnect the barrel from the container. What must the ratio of pressure of gas in the source to initial pressure in the container be if we want to repeat this process exactly 13 times in order to increase the pressure in the container to exactly 13 times its initial value? Assume that the ambient temperature is constant and that both the barrel and the container are in thermal contact with their surroundings. *Jáchyma got an idea on Friday the 13th.*

After connecting a barrel with volume V_b to the source, the pressure in the barrel will become equal to the pressure of the source p_z. In the container with volume V_n, we have gas with pressure

p_0. After connecting the barrel to the container, the pressure in both vessels will become p_1. Since everything happens at constant temperature, it follows from the equation of state that

$$p_1\left(V_\mathrm{b}+V_\mathrm{n}\right)=p_\mathrm{z}V_\mathrm{b}+p_0V_\mathrm{n}\,.$$

We can write this in the form $p_1=k\,p_0+(1-k)\,p_\mathrm{z}$, where we used the substitution

$$k=\frac{V_\mathrm{n}}{V_\mathrm{n}+V_\mathrm{b}}\,. \tag{3}$$

We found an equation describing the change in pressure in the container after connecting and disconnecting the barrel once. If the whole process happens n times, the pressure in the container will be

$$p_\mathrm{n}=k^np_0+(1-k)\,p_\mathrm{z}\sum_{i=1}^{n}k^{i-1}=k^np_0+(1-k)\,p_\mathrm{z}\frac{k^n-1}{k-1}=k^np_0+(1-k^n)\,p_\mathrm{z}\,.$$

The ratio of pressure p_z to p_0 then satisfies

$$\frac{p_\mathrm{z}}{p_0}=\frac{\frac{p_\mathrm{n}}{p_0}-k^n}{1-k^n}\,.$$

If we substitute 13 for n, the expression in (3) for k, and use conditions from the problem statement for V_n and p_{13}, that is, $V_\mathrm{n}=13V_\mathrm{b}$ and $p_{13}=13p_0$, we can compute the result in the form

$$\frac{p_\mathrm{z}}{p_0}=\frac{13-\left(\frac{13}{14}\right)^{13}}{1-\left(\frac{13}{14}\right)^{13}}\,.$$

Numerically, $p_\mathrm{z}/p_0\doteq 20.40$.

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Problem FoL.49 ... space cowboy

An astronaut with a space gun is floating freely in empty space. The gun is a futuristic weapon, that, when fired, imparts the projectile with kinetic energy $E_\mathrm{k}=1\cdot 10^{17}\,\mathrm{J}$ (measured in the reference frame in which the center of mass of the gun and projectile is at rest). Find the velocity with which the projectile will move away from the astronaut (in the reference frame of the astronaut). The mass of the astronaut with the gun is $m_1=100\,\mathrm{kg}$, the mass of the projectile is $m_2=10\,\mathrm{kg}$. Neglect any decrease in mass due to burning of explosive charges and rotation of objects. ***Enter the result as a multiple of the speed of light (in units of c).***

Mirek thinks that guns are operational in vacuum.

Let's observe the event from the reference frame of the center of mass, in which the astronaut is initially at rest. The kinetic energy of the fired projectile is

$$E_\mathrm{k}=\sqrt{m_2^2c^4+p_2^2c^2}-m_2c^2\,.$$

Next, let's express the momentum of the projectile

$$p_2=\frac{\sqrt{E_\mathrm{k}^2+2E_\mathrm{k}m_2c^2}}{c}\,.$$

In this problem, energy is not conserved, but momentum is, so $\boldsymbol{p}_1 = -\boldsymbol{p}_2$ holds. The momentum can also be expressed in the form $\boldsymbol{p}_i = \gamma_i m_i \boldsymbol{v}_i$, where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v_i^2}{c^2}}} \,.$$

We get an equation for the velocity of the projectile

$$\frac{\sqrt{E_\mathrm{k}^2 + 2E_\mathrm{k} m_2 c^2}}{c} = \frac{1}{\sqrt{1 - \frac{v_2^2}{c^2}}} m_2 v_2 \,. \tag{4}$$

We can express the velocity of the projectile as

$$v_2 = c\frac{\sqrt{\varepsilon^2 + 2\varepsilon}}{1 + \varepsilon} \,,$$

where $\varepsilon = E_\mathrm{k}/(m_2 c^2)$ is the ratio of kinetic energy to rest energy of the projectile. To compute the velocity of the astronaut, we'll use equation (4) too, we only need to change the indices on the right hand side from 2 to 1 (and watch out for the sign – the astronaut and the projectile are moving in opposite directions). We find the expression

$$v_1 = c\sqrt{\frac{\varepsilon^2 + 2\varepsilon}{\mu^2 + 2\varepsilon + \varepsilon^2}} \,,$$

where $\mu = m_1/m_2$.

Now we just need to combine the velocities correctly. Our center-of-mass observer is moving with velocity v_1 with respect to the astronaut and the projectile is moving with velocity v_2 with respect to the observer. From the astronaut's point of view, using the relativistic velocity addition formula, the projectile will move with velocity

$$u_2 = \frac{v_2 + v_1}{1 + \frac{v_1 v_2}{c^2}} \,.$$

After plugging it all in, the result is numerically $u_2 \doteq 0.4743c$.

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Problem FoL.50 ... pressure cooking reloaded

A pressure cooker is a closed pot with walls of thickness $t = 5\,\mathrm{mm}$, thermal conductivity $\lambda = 9\,\mathrm{W{\cdot}m^{-1}{\cdot}K^{-1}}$, volume $V = 3\,\mathrm{l}$ and surface area $S = 4\,\mathrm{dm^2}$. There is a single hole in its walls with diameter $d = 4\,\mathrm{mm}$. We start heating the pot with power $P = 7\,\mathrm{kW}$, pour $V_\mathrm{v} = 1\,\mathrm{l}$ of water inside close the lid and bring it to boil. The room temperature is $T_\mathrm{i} = 20\,^\circ\mathrm{C}$. Compute the increase in the boiling point of water (in °C) with respect to the boiling point at standard conditions (in an open pot), which is 100 °C. *Xellos knows how to cook water.*

Let's use the result of the previous version of this problem: the pressure inside the pot is given by the formula

$$p = p_\mathrm{a} + \frac{RT}{Mp}\frac{8}{\pi^2 d^4 l^2}\left(P - \frac{\lambda S}{t}(T - T_\mathrm{i})\right)^2 \,.$$

We still know neither the pressure nor the temperature in the pot. The pressure depends on the boiling point according to Clausius-Clapeyron equation

$$p = p_\mathrm{a} \exp\left(-\frac{L}{R}\left(\frac{1}{T} - \frac{1}{T_\mathrm{v}}\right)\right) ,$$

where $T_\mathrm{v} = 100\,^\circ\mathrm{C}$ is the boiling point at atmospheric pressure and $L = lM$ is the molar latent heat of evaporation of water. We could combine these equations, but we'd get a nasty equation for T which we could only solve numerically. We can, however, use the fact that the pressure and boiling point won't change much inside the pot. Let's denote $\Delta T = T - T_\mathrm{v}$ and expand

$$\frac{p}{p_\mathrm{a}} \approx 1 - \frac{L}{R}\left(\frac{1}{T} - \frac{1}{T_\mathrm{v}}\right) \approx 1 + \frac{L}{R}\frac{\Delta T}{T_\mathrm{v}^2} ,$$

$$\frac{lM}{R}\frac{\Delta T}{T_\mathrm{v}^2} \approx \frac{8RT_\mathrm{v}}{\pi^2 d^4 l^2 M p_\mathrm{a}}\left(P - \frac{\lambda S}{t}(T_\mathrm{v} - T_\mathrm{i})\right)^2 ,$$

$$\Delta T \approx \frac{8R^2 T_\mathrm{v}^3}{\pi^2 d^4 l^3 M^2 p_\mathrm{a}^2}\left(P - \frac{\lambda S}{t}(T_\mathrm{v} - T_\mathrm{i})\right)^2 .$$

We utilised the approximations $T \approx T_\mathrm{v}$, $p \approx p_\mathrm{a}$. We get $\Delta T \doteq 0.46\,^\circ\mathrm{C}$ – it is clear that the assumption $T \approx T_\mathrm{v}$ holds. Since the change in the boiling point is sufficiently small, we'd reach a very similar result ($0.45\,^\circ\mathrm{C}$) by directly using the pressure computed in the previous version of this problem and the Clausius-Clapeyron equation.

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Problem FoL.51 ... balls of steel

There are two conducting balls with radii $R_1 = 0.1\,\mathrm{m}$, $R_2 = 0.2\,\mathrm{m}$ inside a homogeneous electric field $\boldsymbol{E} = E\boldsymbol{e}_z$, where $E = 50\,\mathrm{kV{\cdot}m^{-1}}$. The spherical coordinates of the center of the second ball with respect to the center of the first one are (r, ϑ, φ), where the vector corresponding to $\vartheta = 0$ lies in the direction of cartesian axis z and $0 \leq \varphi < 360^\circ$. Find the force between those balls for $R_{1,2} \ll r = 5\,\mathrm{m}$, $\vartheta = 30^\circ$, $\varphi = 50^\circ$. *Xellos wanted to make a problem that has balls.*

Let us begin with only one ball R in a homogeneous field. The ball is a conductor, so there will be induced charge on its surface distributed in such a way that the resulting electric potential on the ball will be constant. In our case this means that there is a linear dependence between the potential of the induced charge and the coordinate z.

This condition is fulfilled by the dipole potential. Two charges $\pm q$ placed along the z axis at a mutual distance $d \ll R$, symetrically with respect to the center of the ball, create a dipole potential

$$V_\mathrm{i}(\boldsymbol{r}) \approx \frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{\sqrt{r^2 + zd}} + \frac{1}{\sqrt{r^2 - zd}}\right) \approx \frac{qzd}{4\pi\varepsilon_0 r^3} .$$

On the surface of the ball, we have $r = R$; the potential of the ambient field is $V_\mathrm{e} = -Ez$ (plus a constant), so we need to create a dipole moment

$$p = qd = 4\pi\varepsilon_0 R^3 E .$$

With two balls, we would have to account for the mutual induction between those two balls. In our case, $R_{1,2} \ll r$, thus we can introduce an approximation that neglects this effect.

Now we have to find the force between two (induced) dipoles. We know that the resulting force will be independent of φ due to symmetry. The dipole moments are

$$\boldsymbol{p}_{1,2} = q_{1,2} d_{1,2} \boldsymbol{e}_z = 4\pi\varepsilon_0 R_{1,2}^3 E\,.$$

The formula for the force between two dipoles is well known:[1]

$$\boldsymbol{F} = \frac{3}{4\pi\varepsilon_0 r^5}\left((\boldsymbol{p}_1\cdot\boldsymbol{r})\boldsymbol{p}_2 + (\boldsymbol{p}_2\cdot\boldsymbol{r})\boldsymbol{p}_1 + (\boldsymbol{p}_1\cdot\boldsymbol{p}_2)\boldsymbol{r} - 5\frac{(\boldsymbol{p}_1\cdot\boldsymbol{r})(\boldsymbol{p}_2\cdot\boldsymbol{r})}{r^2}\boldsymbol{r}\right)\,.$$

Since $\boldsymbol{p}_1, \boldsymbol{p}_2$ point in the direction of z and $z = r\cos\vartheta$, we obtain

$$\boldsymbol{F} = \frac{3p_1p_2}{4\pi\varepsilon_0 r^5}\left(2r\cos\vartheta\boldsymbol{e}_z + (1-5\cos^2\vartheta)\boldsymbol{r}\right)\,.$$

The expression in parentheses has a perpendicular component $(1-5\cos^2\vartheta)r\sin\vartheta$ and parallel component $(3-5\cos^2\vartheta)r\cos\vartheta$. Its absolute value then is, by the Pythagorean theorem,

$$r\sqrt{1+5\cos^4\vartheta-2\cos^2\vartheta} = r\sqrt{\sin^4\vartheta+4\cos^4\vartheta}$$

and the resulting force is

$$F = \frac{3p_1p_2}{4\pi\varepsilon_0 r^4}\sqrt{\sin^4\vartheta+4\cos^4\vartheta} = \frac{12\pi\varepsilon_0 R_1^3R_2^3E^2}{r^4}\sqrt{\sin^4\vartheta+4\cos^4\vartheta} \doteq 1.62\cdot10^{-8}\,\mathrm{N}\,.$$

We can easily show that the dipole field decreases with distance so fast that the mutual induction is negligible. The dipole field of the second ball at the distance r is weaker than the homogeneous field by approximately $(R_2/r)^3 < 1\cdot10^{-4}$, a similar evaluation can be done for the first ball.

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Problem M.1 ... tic-toc

How many times greater is the distance travelled by the tip of the second hand on a clock face than the distance travelled by the tip of the minute hand, during one day (exactly 24 hours)? The length of the second hand is 105 mm*, of the minute hand* 100 mm*.*

Kiki was killing some time.

The travelled distance s is given by $s = r\varphi$, where r is the radius and φ is the angle (in radians). The full angle is 2π. Let us see how many full rotations does each hand make during one day. The second hand makes one rotation per minute, making it 1,440 rotations per day. The minute hand makes 24 rotations per day. Since we are interested in the ratio of distances, factor 2π cancels out and the result is

$$x = \frac{1{,}440 r_1}{24 r_2}\,.$$

For given values we get $x = 63$. The second hand will travel 63 times greater distance than the minute hand.

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[1] The derivation is not hard, but it is onerous. For two magnetic dipoles, the formula can be found e. g. on Wikipedia `https://en.wikipedia.org/wiki/Magnetic_dipole#Forces_between_two_magnetic_dipoles`. To express the force between electric dipoles, we just need to make substitutions $\boldsymbol{m} \leftrightarrow \boldsymbol{p}$ and $\mu_0 \leftrightarrow 1/\varepsilon_0$.

Problem M.2 ... far from school

Luboš and Náry leave the student dormitory at the same time and both head (separately) to school. The distance between the school and the dormitory is $l = 350\,\mathrm{m}$. Luboš has got a swift pace, $v_\mathrm{L} = 2\,\mathrm{m{\cdot}s^{-1}}$, but he is also very forgetful. During his walk to school he had to return three times to the dormitory – the turning points were at distances $100\,\mathrm{m}$, $200\,\mathrm{m}$ and $300\,\mathrm{m}$ from the dormitory. Each time he spent $t_\mathrm{L} = 1\,\mathrm{min}$ inside the dormitory. Náry walks at the same speed as Luboš, $v_\mathrm{N} = v_\mathrm{L}$, but he doesn't turn back to get the things he forgot. However, he has another bad habit – stopping and talking to passersby. Let's assume he bumps into someone every $50\,\mathrm{m}$ (including the dormitory entrance and school entrance) and talks to him for $t_\mathrm{N} = 1.5\,\mathrm{min}$. What will be the difference of Luboš's and Náry's travel times? If Náry arrives first, give the result as a negative number. *Mirek counted how many times Luboš returned.*

Luboš returns three times, every time walking the same distance ($100\,\mathrm{m}$, $200\,\mathrm{m}$ and $300\,\mathrm{m}$) twice, and then he walks straight from dormitory to school ($350\,\mathrm{m}$), so the total travel time is

$$T_\mathrm{L} = 3t_\mathrm{L} + \frac{2}{v_\mathrm{L}}\left(100\,\mathrm{m} + 200\,\mathrm{m} + 300\,\mathrm{m}\right) + \frac{350\,\mathrm{m}}{v_\mathrm{L}} \doteq 955\,\mathrm{s}\,.$$

Náry stops eight times, but walks without returning, so his travel time is

$$T_\mathrm{N} = 8t_\mathrm{N} + \frac{350\,\mathrm{m}}{v_\mathrm{N}} \doteq 895\,\mathrm{s}\,.$$

The difference then is $T_\mathrm{N} - T_\mathrm{L} \doteq -60\,\mathrm{s}$, meaning Luboš will arrive one minute after Náry.

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Problem M.3 ... not falling well

A cylindrical glass with radius of the base $R = 5\,\mathrm{cm}$ is filled with beer so that the surface of the liquid reaches $h = 4\,\mathrm{mm}$ above the edge (measured at the center of the surface). This is due to the strong surface tension of $\sigma = 72\,\mathrm{mN{\cdot}m^{-1}}$. A fly just drowned in the beer and floats at the distance $r = 2\,\mathrm{mm}$ from the center of the surface. How fast will the fly be moving when it reaches the edge of the glass? Assume it was initially at rest, the slope of the surface close to the center is small (but non-zero) and all resistive forces are negligible.

Xellos was drinking a beer.

This is an easy problem relying on the law of energy conservation – velocity is independent of the trajectory and is related to the height difference Δh through the formula

$$v = \sqrt{2g\Delta h}\,.$$

Since we assumed the surface is almost level near the center, the fly must have been initially in the height h above the edge of the glass. Right before reaching the edge of the glass, its height will be 0, therefore $\Delta h = h$ and $v \approx \sqrt{2gh} = 0.28\,\mathrm{m{\cdot}s^{-1}}$.

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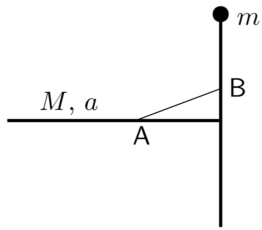
Problem M.4 ... don't fall at all



In the figure there is a construction made of two rods as seen from above. The length and mass of each rod are a and M. A mass $m = M/4$ is attached to the end of one of the rods. We have welded another homogeneous rod to the construction (the thin line in the figure) so that the center of mass of the new rod coincides with the center of mass of the construction. Find the length of the third rod (distance between A and B in the figure) and give the result as a multiple of a.

Mirek was unhappy because he couldn't touch the center of mass.

First, let us find the coordinates of the center of mass T of the construction

$$x_{\mathrm{T}} = \frac{M\frac{a}{2} + Ma + \frac{M}{4}a}{\frac{9}{4}M} = \frac{7}{9}a\,,$$

$$y_{\mathrm{T}} = \frac{\frac{M}{4}\frac{a}{2}}{\frac{9}{4}M} = \frac{1}{18}a\,,$$

where the origin was put at the left end of the left-right pointing rod. The new rod must be welded on the construction in such a way that the distance between A and the center of mass is equal to the distance between B and the center of mass. So

$$|AB| = \sqrt{(2(a - x_{\mathrm{T}}))^2 + (2y_{\mathrm{T}})^2} = \frac{\sqrt{17}}{9}a\,.$$

In multiples of a, the length of the new rod is approximately 0.46.

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Problem E.1 ... breaking a capacitor

We've got a capacitor formed by two separated conducting plates. We charge it and disconnect it from the voltage source. How many times larger will the charge we can get from the capacitor be, if we break it into four identical pieces and place them on top of each other without rotating them in any way? *Štěpán was breaking chocolate into pieces.*

Let's denote the capacity of the original capacitor by C_0 and its voltage by U. If we split it into 4 identical pieces, we get capacitors with quarter capacities (since their surfaces decrease to a quarter of the original) and the same voltage U.

If we combine them in series, the total capacity will be $C_0/16$ (you can verify this yourselves) and the voltage will be 4 times larger.

We can compute the total charge as

$$Q = 4U\frac{1}{16}C_0 = \frac{1}{4}UC_0 = \frac{1}{4}Q_0\,.$$

The charge will be four times smaller.

We can reach the same result by working with capacitor energy, which must be conserved.

Note that the problem can be solved in an even simpler way. We only need to notice that the pieces of plates will have only a quarter of the original charge, so the total charge we can obtain will also be just a quarter of the original.

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Problem E.2 ... ideal voltage

Consider an ideal voltage source. First, we connect to it in parallel three identical resistors and find out that the total power consumed by the resistors is P. What will be the consumed power if we connect the resistors to the source in series? Compute the result as a multiple of P. *Karel was teaching about electric voltage.*

When the resistors with identical resistances R are connected in parallel, their total resistance is equal to $R/3$. When they are connected in series, it's $3R$ (you can verify that yourselves). In order to compute the power P of resistors connected in parallel, we'll use the well-known formula for power consumed by a resistor and Ohm's law. We get

$$P = UI = \frac{U^2}{R_c} = \frac{3U^2}{R}\,,$$

where I is the total current flowing through the circuit, U is the voltage of the source and R_c is the total resistance of the circuit. The power P' of resistors connected in series is

$$P' = UI = \frac{U^2}{R_c} = \frac{U^2}{3R}\,.$$

Simple division now yields $P' = \frac{1}{9}P$.

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Problem E.3 ... ideal current

Consider an ideal current source. First, we connect to it three identical resistors in parallel and find that the total power consumed by the resistors is P. What will the consumed power be if we connect the resistors to the source in series? Compute the result as a multiple of P.
Karel was teaching about electric current.

When the resistors with identical resistances R are connected in parallel, their total resistance is equal to $R/3$. When they are connected in series, it's $3R$ (you can verify that yourselves). In order to compute the power P of resistors connected in parallel, we'll use the well-known formula for power consumed by a resistor and Ohm's law. We get

$$P = UI = R_c I^2 = \frac{1}{3}RI^2\,,$$

where I is the total current flowing through the circuit, U is the voltage of the source and R_c is the total resistance of the circuit. The power P' of resistors connected in series is

$$P' = UI = R_c I^2 = 3RI^2\,.$$

Simple division now yields $P' = 9P$.

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Problem E.4 ... loaded triangle

We built a triangle from three resistors with equal resistances and connected voltage sources parallel to two of the resistors in such a way that their negative poles are connected to the same vertex. How many times larger will the current flowing through the source with voltage $7U$ be compared to the current flowing through the other source with voltage $5U$?

Kuba forced Štěpán to make some problems instead of programming.

First, let's perform the triangle-star transformation. We only need to realise that all resistances are identical, so the resistors in the star will also be identical; let's denote it by R.

We can now apply Kirchhoff's laws. Let's denote the current flowing through the source with smaller voltage by I_1, the current flowing through the larger source by I_2 and the current through the last resistor by I_3. We assume that the currents flow through the sources in the standard direction (we aren't charging them). Then, we get the equation $I_1 + I_2 = I_3$. (We could get slightly different equations if we defined the currents in different directions, but the final result will always be the same.)

For the first loop, we get $I_1R + I_3R = 5U$ and for the second one, $I_2R + I_3R = 7U$. Solving these three equations with three unknown variables, we find that $I_2 = 3U/R$ and $I_1 = U/R$, so the current flowing through the larger source is 3 times larger.

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Problem X.1 ... a nice cup of tea

Somebody mixed the polonium-210 isotope (half-life 138 days*) into Mikuláš's tea. Luckily, Mikuláš noticed that. However, he really wants some tea, so he measured the amount of radionuclide in his tea and calculated when will the radioactivity decrease to a safe-to-drink value. After this time of* 342 days *he drank his tea with no side effects. A few years later, somebody mixed twice the original amount of polonium-210 into his tea. How long does Mikuláš have to wait this time?*

Guys, I really am not a retired Russian agent...

All we need is to recall the definition of half-life. Then it is clear that if we wait 138 days, the new problem will reduce itself to the old one (i. e. the amount of the radionuclide will decrease to the original value). Therefore, the waiting time will be $138 + 342 = 480$ days.

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Problem X.2 ... cheesy

Vítek found a shop with suspiciously cheap cheese. On Monday at 14:15, he bought 250 g *of cheese there. On Tuesday at 17:21, he bought another* 200 g *and on Wednesday at 19:45, yet another* 160 g*. He always stored the cheese he bought carefully in the fridge. Unfortunately, he hadn't noticed that the cheese contained the radioactive nuclide* ^{24}Na *with half-life* 15 hrs.

The mass fraction of this substance in the cheese at the moment when the cheese is bought is $w = 0.0013$. What will be the mass (in grams) of non-decayed ^{24}Na in the cheese at the moment when Vítek wants to eat it – on Thursday at 9:11? *Jáchym was skipping through sale flyers.*

Radioactive decay follows the equation

$$m' = m\mathrm{e}^{-\frac{t}{T}\ln 2}\,,$$

where m is the initial mass of the isotope and m' is its mass at time t. The last variable in the equation is half-life T.

Let's denote the time period between the first sale and eating the cheese by t_1 and the mass of bought cheese by M_1. The initial mass of the isotope in cheese is $m_1 = M_1 w$. The mass of the isotope after time t_1 then is

$$m_1' = m_1\mathrm{e}^{-\frac{t_1}{T}\ln 2} = M_1 w\mathrm{e}^{-\frac{t_1}{T}\ln 2}\,.$$

Similarly, we can compute the masses of the isotope in the other two pieces of cheese, which we'll denote by m_2' and m_3'. The resulting mass is the sum of masses in individual pieces, which is

$$m_1' + m_2' + m_3' \doteq 0.168\,\mathrm{g}\,.$$

The mass of the non-decayed sodium isotope is 168 mg.

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Problem X.3 ... yoghurt

Jáchym eats half a kilogram of yoghurt for dinner every day. The yoghurt contains a mass fraction $w = 10^{-4}$ of a certain radioactive isotope with half-life 26 days. How many grams of this isotope will be present in Jáchym's body every day before dinner, if he's been eating this way for a long time? Assume that the isotope doesn't leave his body in any way other than through radioactive decay. *Jáchym, eating a yoghurt.*

Right before dinner, Jáchym contains mass m_0 of this isotope. The yoghurt contains mass $\Delta m = w m_\mathrm{j}$ of the isotope, where m_j is the mass of the yoghurt. After Jáchym eats the yoghurt, the mass of the isotope he contains increases to $m_1 = m_0 + \Delta m$. During the following 24 hours, this mass gradually decreases to the original value m_0. We can use the formula for radioactive decay

$$m_0 = m_1\mathrm{e}^{-\lambda t}\,.$$

Substituting for m_1, we get

$$m_0 = (m_0 + \Delta m)\,\mathrm{e}^{-\lambda t}\,.$$

Now, we can express

$$m_0 = \Delta m \frac{\mathrm{e}^{-\lambda t}}{1 - \mathrm{e}^{-\lambda t}}\,.$$

The time period t has length 1 day and the radioactive decay constant λ satisfies

$$\lambda = \frac{\ln 2}{t_{\frac{1}{2}}}\,,$$

where $t_{\frac{1}{2}}$ is the half-life. After plugging in all the numbers, we get $m_0 \doteq 1.85\,\mathrm{g}$.

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Problem X.4 ... beer

The average volume ratio of air to beer in beer foam is 6.87*. The half-life of an average bubble is* 73 s*. If, initially, the beer reaches the height* 13 cm *above the bottom of the glass and the foam reaches height* 16.5 cm*, what height (in centimetres) will the foam reach after two minutes?*

Made up by Jáchym while watching a Czech cult film.

The beer reaches the height h_1 and the foam h_2. The initial height of foam is $h_0 = h_2 - h_1$. The decay of bubbles will proceed in the same way as radioactive decay – the height of foam at time t is

$$h(t) = h_0 \mathrm{e}^{-\frac{t}{T}\ln 2}\,,$$

where T denotes the half-life. The beer and air released by this decay will have height h_p and h_v respectively. We obtain the equation

$$\begin{aligned} h_\mathrm{p} + h_\mathrm{v} &= h_0 - h\,,\\ 1 + \frac{h_\mathrm{v}}{h_\mathrm{p}} &= \frac{h_0 - h}{h_\mathrm{p}}\,,\\ h_\mathrm{p} &= \frac{h_0 - h}{k+1}\,, \end{aligned}$$

where $k = h_\mathrm{v}/h_\mathrm{p}$ is the given ratio of air to beer. The height which the foam will reach in the glass can be computed as

$$H = h_1 + h_\mathrm{p} + h = h_1 + (h_2 - h_1)\,\frac{k\mathrm{e}^{-\frac{t}{T}\ln 2} + 1}{k+1} \doteq 14.42\,\mathrm{cm}\,.$$

The foam will reach 14.42 cm from the bottom of the glass.

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