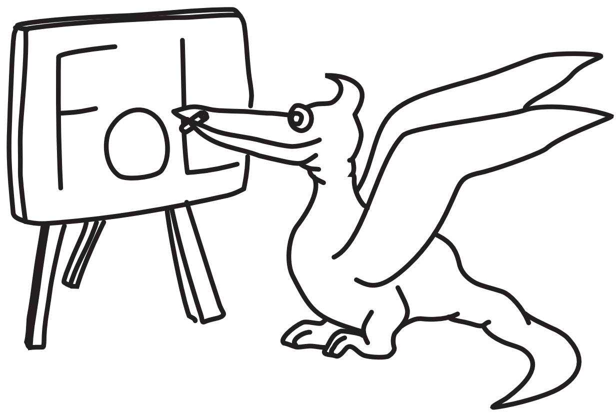
Solutions of 5th Online Physics Brawl



Problem FoL.1 ... misbehaved students

On a planet far, far away, some misbehaved students threw an object out of a window, located $h = 40\,\mathrm{m}$ above the ground. It landed after $t = 4\,\mathrm{s}$. How much would this object weigh on Earth, if the gravitational force acting upon the object on the distant planet is $F = 55\,\mathrm{N}$? Consider the gravitational field to be constant on both planets and the air drag to be negligible.

Olda was having plenty of time while Náry was cooking.

To solve this problem, we must be aware of the difference between the weight and the mass of an object, i.e. that its mass is independent of its location and the intensity of the gravitational field. Our first task is to find the gravitational acceleration on the distant planet, and then, using Newton's second law, to get the mass of the object (which is equal to its weight on Earth divided by g).

The equation for free fall is

$$h = \frac{1}{2}at^2\,,$$

which we can rearrange to express the acceleration

$$a = \frac{2h}{t^2} = 5\,\mathrm{m{\cdot}s^{-2}}\,.$$

To find the weight, we divide the force $F = 55\,\mathrm{N}$ by the gravitational acceleration

$$m = \frac{F}{a} = \frac{Ft^2}{2h} = 11\,\mathrm{kg}\,.$$

The object weighs 11 kg.

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Problem FoL.2 ... plastered block

Kuba is boasting that he can lay a perfectly smooth cuboid on an inclined plank in such a way that the cuboid won't slip down. The plank is inclined at an angle $\alpha = 20^\circ$ to the horizontal plane. Find out what Kuba's acceleration has to be when he holds the plank, so that the cuboid would remain at rest (with respect to the plank). *Mirek doesn't believe such silly tricks...*

It's a straightforward analysis of forces. The cuboid is pulled down by gravity mg, where m is its mass and g is the acceleration due to gravity. When projected onto the plank, it's $mg\sin\alpha$. Kuba's acceleration $\boldsymbol{a}$ will result in a reaction force ma in the opposite direction to Kuba's acceleration. Its projection onto the plank is $ma\cos\alpha$. The cuboid won't move if both projections are equal in magnitude

$$mg\sin\alpha = ma\cos\alpha\,,$$

from which we can express

$$a = g\,\mathrm{tg}\,\alpha\,.$$

After substitution $g = 9.8\,\mathrm{m{\cdot}s^{-2}}$, we get a numeric result $a = 3.6\,\mathrm{m{\cdot}s^{-2}}$.

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Problem FoL.3 ... we're on a roll

A truncated cone is rolling on a table in such a way, that a point on the perimeter of its smaller base moves at a speed $v_1 = 1.5\,\mathrm{m\cdot s^{-1}}$ with respect to the base's center and a point on the perimeter of its greater base moves at a speed $v_2 = 1\,\mathrm{m\cdot s^{-1}}$ (again, with respect to the base's center). The slant height is $l = 0.1\,\mathrm{m}$. How long will it take the cone to return to the point from which it started rolling? *Tom dropped a pestle.*

The greater base's point of contact with the table follows a circle of a radius $R = l + L$, where L is the slant height of a cone which we would have to cut off from a normal cone in order to obtain the truncated one. Both bases move at the same angular velocity, thus

$$\frac{v_1}{r_1} = \frac{v_2}{r_2}\,,$$

from which we isolate

$$\frac{r_1}{r_2} = \frac{v_1}{v_2}\,.$$

Then, from the similarity of triangles, we have

$$\frac{r_2}{L} = \frac{r_1}{L+l}\,,$$

so

$$1 + \frac{l}{L} = \frac{r_1}{r_2} = \frac{v_1}{v_2}\,.$$

For the unknown variable L, we can write

$$L = \frac{l}{\frac{v_1}{v_2} - 1}\,.$$

The truncated cone returns to the starting point when the greater base makes a whole circle of the radius $R = L + l$, so the time is

$$t = \frac{2\pi(L+l)}{v_1} = \frac{2\pi l}{v_1}\left(1 + \frac{1}{\frac{v_1}{v_2} - 1}\right) \doteq 1.26\,\mathrm{s}\,.$$

It takes the cone $t \doteq 1.26\,\mathrm{s}$ to return to the point from which it started rolling.

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Problem FoL.4 ... cooling

Consider a Peltier cell with cooling power $P = 10\,\mathrm{W}$, placed on a person's wrist. Let's assume that all of its power is spent on cooling blood flowing through a vein with volumetric flow rate $Q = 1.6\,\mathrm{ml\cdot s^{-1}}$. Blood density is $\varrho = 1\,025\,\mathrm{kg\cdot m^{-3}}$, its heat capacity $c = 4.2\,\mathrm{kJ\cdot kg^{-1}\cdot K^{-1}}$. How much cooler is the blood after flowing through the wrist?

Filip had to improvise in the summer.

The mass flow rate of blood through the vein is ϱQ. Let's express the cooling power as heat taken away over a time element Δt,

$$c\varrho Q \Delta t = P\,,$$
$$\Delta t = \frac{P}{c\varrho Q}\,,$$

which leads to $\Delta t \doteq 1.5\,^{\circ}\mathrm{C}$.

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Problem FoL.5 ... wet road

How much does the shortest braking distance increase when it rains, if we do not want the car to skid? The car is moving at a speed of $v = 90\,\mathrm{km{\cdot}h^{-1}}$, the coefficient of friction between a dry road and the tires is $f_1 = 0.55$ and it changes to $f_2 = 0.30$ in wet conditions. The weight of the car is $m = 1\,500\,\mathrm{kg}$ and the acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. Consider only kinetic energy of the car's translational motion. *Pikoš barely managed to stop.*

For the car to not skid, the braking force mustn't exceed the friction force, which is given by $F_\mathrm{t} = fF_\perp$, where $F_\perp$ is the normal force from the car acting upon the road - in our case, it's equal to the force of gravity. Therefore, $F_\perp = mg$, which means $F_\mathrm{t} = fmg$. If this force is exerted over a distance of l, it changes the kinetic energy by $\Delta E = E_\mathrm{k}$.

The kinetic energy of the car in the beginning is $E_\mathrm{k} = mv^2/2$; after the car stops, it's zero, so the work needed to stop the car is $\Delta E = E_\mathrm{k}$, from which we can find the braking distance

$$l = \frac{v^2}{2fg}\,.$$

The difference between the braking distance on a wet and a dry road is then

$$\Delta l = \frac{v^2}{2g}\left(\frac{1}{f_2} - \frac{1}{f_1}\right) \doteq 48.3\,\mathrm{m}\,.$$

The braking distance increases by about 48.3 m, which means it almost doubles.

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Problem FoL.6 ... high five!

Tomáš and Michal have met on a cycleway and want to high five while riding. Both of them have the same mass $m = 70\,\mathrm{kg}$, speed $v = 8\,\mathrm{m{\cdot}s^{-1}}$ and their arms are stretched horizontally at a distance of $l = 1\,\mathrm{m}$ from the centres of mass of their bodies. They are moving along parallel lines, but in opposite directions. The distance between their centres of mass at the moment when they are passing by is exactly $2l$. Determine the frequency (in Hz) with which they would rotate around their common centre if they grabbed each other's hands. Neglect the mass of their bicycles and energy loss during the act of catching each other. Consider the cyclists to be mass points. *Mirek making friends with a birch while riding on a bicycle.*

Since we consider the cyclists to be mass points, it is obvious that their angular momentum and mechanical energy are conserved if the peripheral speed of each cyclist during the rotation corresponds to translational speed at the beginning. Let's denote the initial state as 0 and the final one as 1. The equations for energies and angular momenta are

$$
\begin{aligned}
E_0 &= \frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2\,,\\
E_1 &= \frac{1}{2}I\omega^2 + \frac{1}{2}I\omega^2 = I\omega^2 = ml^2\omega^2\,,\\
L_0 &= 2mvl\,,\\
L_1 &= 2m\omega l^2\,.
\end{aligned}
$$

where we used the definition of the moment of inertia for a mass point $I = ml^2$. The frequency is

$$f = \frac{\omega}{2\pi} = \frac{v}{2\pi l} = 1.3\,\mathrm{Hz}\,.$$

The cyclists will rotate with this frequency until they fall to the ground (which will happen quickly).

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Problem FoL.7 ... (non)broken thermometer

We have a particular thermometer with which we measured the temperature of melting ice as $t_0 = -0.3\,^\circ\mathrm{C}$ and the temperature of water vapor as $t_\mathrm{v} = 101.4\,^\circ\mathrm{C}$. What is the real boiling point τ of methanol at atmospheric pressure, if the thermometer shows a temperature $t = 65.5\,^\circ\mathrm{C}$? Assume that the real temperature of melting ice is $\tau_0 = 0\,^\circ\mathrm{C}$ and of water vapor $\tau_\mathrm{v} = 100\,^\circ\mathrm{C}$. Also assume that the number of ticks on the broken thermometer is proportional to the real temperature in Celsius degrees. *Lydka played with a thermometer during laboratory practice.*

The given values imply that the temperature difference of $100\,^\circ\mathrm{C}$ matches 101.7 ticks on the thermometer. Using the assumptions that the capilary of the thermometer is divided by the ticks into parts of equal volume and that the scale reading is a linear function of the real temperature, we obtain

$$\frac{\tau_\mathrm{v} - \tau_0}{\tau - \tau_0} = \frac{t_\mathrm{v} - t_0}{t - t_0}\,.$$

For $\tau_0 = 0\,^\circ\mathrm{C}$,

$$\tau = \frac{t - \tau_0}{t_\mathrm{v} - t_0}\tau_\mathrm{v} = 64.7\,^\circ\mathrm{C}\,,$$

which is the temperature which a correctly working thermometer would show.

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Problem FoL.8 ... line of apsides

Consider a terrestrial planet which orbits its maternal solar-type star in such a way, that we can neglect other bodies in the universe and the eccentricity the of planet's orbit is $e = 0.150\,0$. Determine the ratio $v_\mathrm{p}/v_\mathrm{a}$ of the speed of the planet at periapsis v_p (the nearest point to the star) to the speed at apoapsis v_a (the farthest point from the star) . Karel exercising.

We use the 2nd Kepler law, which tells us that a line segment joining the planet and the star sweeps out equal areas during equal intervals of time. Hence, we have an equation

$$v_\mathrm{a} r_\mathrm{a} = v_\mathrm{p} r_\mathrm{p}\,, \tag{1}$$

where r_p is the distance from the star to the planet at periapsis and analogously r_a is the distance to the planet at apoapsis. We can understand easily from the geometry of an ellipse that $r_\mathrm{p} = a\,(1-e)$ and $r_\mathrm{a} = a\,(1+e)$, where a is length of the semi-major axis of the ellipse. After plugging this into the equation (1) and rearranging it, we get the ratio

$$\frac{v_\mathrm{p}}{v_\mathrm{a}} = \frac{1+e}{1-e} \doteq 1.353\,.$$

The ratio of speeds is 1.353.

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Problem FoL.9 ... the fire rises

What is the lowest temperature of the air (in °C) inside a balloon, which will make it float? The volume of the balloon is $V = 6\cdot 10^6\,\mathrm{l}$ and its weight (without the air inside) is $m = 700\,\mathrm{kg}$. The density of surrounding air is $\varrho = 1.2\,\mathrm{kg{\cdot}m^{-3}}$ and the atmospheric pressure is $p = 101\,\mathrm{kPa}$. Xellos was baneposting.

For the balloon to float, its density must be equal to the density of the surrounding air. We know the volume of the balloon, so we can find the weight of the air inside it m_v

$$\varrho = \frac{m + m_\mathrm{v}}{V} \quad \Rightarrow \quad m_\mathrm{v} = \varrho V - m = 6.5\,\mathrm{t}\,.$$

Using the ideal gas law, we can find the temperature of the air, knowing its weight (the pressure inside is still equal to the atmospheric pressure)

$$T = \frac{pV}{Rn} = \frac{pVM_\mathrm{v}}{Rm_\mathrm{v}} = \frac{pVM_\mathrm{v}}{R\,(\varrho V - m)} = 325\,\mathrm{K}\,,$$

where $M_\mathrm{v} = 28.97\,\mathrm{g{\cdot}mol^{-1}}$ is the molar mass of air. This temperature is equal to $52\,^\circ\mathrm{C}$.

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Problem FoL.10 ... sunny

Mirek decided to do something for his health, so he took his schoolbooks outdoors for fresh air. The sky was clear and the sun was exactly at the south, $\alpha = 50^\circ$ above the horizon. If Mirek places his book horizontally, then it receives radiation power P from the sun. Determine the lowest angle β by which Mirek must tilt the book towards himself (he looks at it from the north) so that the incident radiation power drops to $P/2$. *Mirek's paper was too white.*

Incident power is proportional to the surface on which light is incident. If a book has an area S_0, then the area of its projection onto a plane perpendicular to the rays is

$$S_1 = S_0 \sin\alpha\,.$$

We seek an area

$$S_2 = S_0 \sin(\alpha - \beta)\,,$$

that satisfies

$$\frac{S_2}{S_1} = \frac{\sin(\alpha-\beta)}{\sin\alpha} = \frac{1}{2}\,.$$

We can either solve this equation numericaly or rewrite it as follows:

$$2\sin\alpha\cos\beta - 2\sin\beta\cos\alpha = \sin\alpha\,.$$

Now we choose $\sin\beta$ as the unknown variable; we get the sines over to the right side, square both sides of the equation, using the Pythagorean theorem we substitute cosines for sines and divide the equation by $cos^2\alpha$, transforming $sin^2\alpha$ to $tg^2\alpha$. After these operations, we get

$$(1 + \operatorname{tg}^2\alpha)\sin^2\beta + \operatorname{tg}\alpha\sin\beta - \frac{3}{4}\operatorname{tg}^2\alpha = 0\,.$$

We seek the lowest possible angle. Therefore, we choose the positive root of the above equation, which means that further tilting the book decreases the angle at which the incoming rays are incident on the book. The result is

$$\sin\beta = 0.461\,4\ldots \quad\Rightarrow\quad \beta \doteq 27^\circ\,.$$

The book must be tilted approximately by 27°.

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Problem FoL.11 ... the potassium foil

When Kiki was young and used to not know anything about quantum physics (she only knew the classical view that energy is delivered continuously by a light beam), her home was lit with an isotropic light source with an output of $P = 1.5\,\mathrm{W}$. At a distance $R = 3.5\,\mathrm{m}$, she placed a foil made of potassium with work function $\varphi = 2.2\,\mathrm{eV}$. Help Kiki determine how many seconds it'd take for the foil to absorb enough energy to emit an electron. Assume that the foil absorbs all the incident energy, and that an emitted electron absorbs the energy incident on a circular surface of radius $r = 5\cdot 10^{-11}\,\mathrm{m}$, given by the typical radius of an atom.

Dominika gazed into the past.

Let's assume that the energy emitted by the source is distrbuted evenly in an expanding spherical wavefront centered at the source. The intensity at a distance R from a point source is given as

$$I = \frac{P}{4\pi R^2}\,.$$

If light of intensity I is incident on a surface $S = \pi r^2$ over time t, an energy $E = ISt$ is absorbed. If only one electron recieves all the 2.2 eV of energy, it needs to absorb it over time

$$t = \frac{\varphi}{IS} = \frac{4\pi\varphi R^2}{PS} \doteq 4\,600\,\mathrm{s}\,.$$

In fact, the emission of electrons doesn't have to occur, since we don't know the energy (wavelength) of incident photons.

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Problem FoL.12 ... tuned

A source plays a tone with frequency $f_0 = 440\,\mathrm{Hz}$ (a'). What is the speed of a car moving straight towards the source, if the driver hears b' flat ($f = 466\,\mathrm{Hz}$)? The speed of sound in the air is $v = 343\,\mathrm{m\cdot s^{-1}}$, the effects of the material of the car can be neglected.

Meggy wanted a musical problem.

According to Doppler's law for a stationary source,

$$f = f_0 \frac{v + v_\mathrm{o}}{v}\,,$$

where v_o is the observer's speed. Expressing the speed, we get

$$v_\mathrm{o} = v\left(\frac{f}{f_0} - 1\right)\,,$$

which gives us the numerical value $v_\mathrm{o} \doteq 20.3\,\mathrm{m\cdot s^{-1}}$.

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Problem FoL.13 ... pitch-black

The retina of the human eye is the most sensitive to the yellow-green colour $\lambda = 550\,\mathrm{nm}$, for which the sensitivity threshold is $P = 1.7\cdot 10^{-18}\,\mathrm{W}$. What is the minimal number of photons that have to hit the retina over one second for the light to be percepted?

Verča had total visual blackout.

The energy of one photon is $E = hf$, which (since we know the wavelength) can be rewritten as $E = hc/\lambda$. For the light to be percepted, the number of photons N that hit the retina in $t = 1\,\mathrm{s}$ has to satisfy the following equation:

$$P = N\frac{E}{t} = N\frac{hc}{\lambda t}\,.$$

From that, we can find the formula for N:

$$N = \left\lceil \frac{P\lambda t}{hc} \right\rceil \doteq 5\,.$$

We can see that 5 photons are enough to stimulate the retina.[1]

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Problem FoL.14 ... pyramid

A school model of a regular four-sided pyramid was made of wood with mass density $\varrho_1 = 600\,\mathrm{kg\cdot m^{-3}}$*. The pyramid had mass* $m_1 = 300\,\mathrm{g}$ *and the ratio between the length of its base edge and the height from the base to the apex was* $2:3$*. One day, naughty students broke into physics room and sawed off the poor pyramid's apex. The section was made parallel to the base. The teacher made a new apex with the same size for the pyramid; however, it was made of wood with mass density* $\varrho_2 = 900\,\mathrm{kg\cdot m^{-3}}$ *and then joined with the truncated pyramid. Although the pyramid looks the same now, its mass is* $m_2 = 309.6\,\mathrm{g}$*. Determine the distance from the base (in* cm*) at which the section was made.* *Meggy remembered solid geometry.*

Let us convert the mass densities into $\mathrm{g\cdot cm^{-3}}$, so $\varrho_1 = 0.6\,\mathrm{g\cdot cm^{-3}}$ and $\varrho_2 = 0.9\,\mathrm{g\cdot cm^{-3}}$.

We can easily compute the volume of the pyramid from its mass and mass density.

$$V_1 = \frac{m_1}{\varrho_1} = 500\,\mathrm{cm^3}\,.$$

Let's denote the length of the base edge by a and its height by v. The latter can be expressed from their ratio

$$v = \frac{3}{2}a$$

and plugged into the formula for the volume of a pyramid

$$V_1 = \frac{1}{3}a^2 v = \frac{a^3}{2}\,.$$

We get the length of the base edge

$$a = \sqrt[3]{2V_1} = 10\,\mathrm{cm}$$

and from that, we easily get that $v = 15\,\mathrm{cm}$.

Let us denote the mass of the cut off apex by m' and the mass of the added apex by m''. The volumes of both are the same, denoted by V'. We know the difference of the two complete pyramids' masses (before the section and after adding the new apex) and therefore, we also know the difference between the masses of the apex parts

$$m_2 - m_1 = m'' - m' = 9.6\,\mathrm{g}\,.$$

We can express the masses again using volumes and mass densities

$$\varrho_2 V' - \varrho_1 V' = 9.6\,\mathrm{g}\,.$$

[1] The number of photons can only be a natural number, that's why we have to take the ceiling function.

When we substitute for the mass densities, we get

$$V' = 32\,\mathrm{cm}^3\,.$$

Since the cut off apex and the pyramid itself are similar, the apex base edge is denoted by a' and its height by v', we can do what we have already done for the whole pyramid and get $a' = 4\,\mathrm{cm}$ a $v' = 6\,\mathrm{cm}$.

Hence, the section was done at the height $v - v' = (15 - 6)\,\mathrm{cm} = 9.0\,\mathrm{cm}$ from the base of the complete pyramid.

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Problem FoL.15 ... battering ram vol. 2

As a part of their offensive, the orcs of Mordor want to break through the gate of Minas Tirith. In a parallel-universe Middle-earth, no progress is made even after calling for Grond, a fire spitting battering ram. Feeling hopeless, Sauron eventually decides to prop up another battering ram horizontally between the gate and a newly built perfectly rigid wall. He then orders to heat the ram so as to exploit thermal expansion properties of its material to break the gate. Find out by how much (in Kelvins) the ram needs to be heated. Assume that the ram has the shape of a cylinder with axis perpendicular to the gate. The material of the ram has linear expansion coefficient $\alpha = 1.2 \cdot 10^{-5}\,\mathrm{K}^{-1}$*, Young's modulus of compression* $E = 211\,\mathrm{GPa}$ *and the maximum compressive stress that the gate endures is* $\sigma_{\max} = 400\,\mathrm{MPa}$*. Consider the thermal expansion process to be fully linear and assume that there is no deformation of the gate until it yields.* *Ondra watched El Señor de los Anillos.*

Obviously, there are two main phenomena affecting the length of the battering ram which need to be considered: thermal expansion and compressive deformation (which is described by Hooke's law). If we denote the relative expansions corresponding to the two above mentioned effects by ε_1 and ε_2 respectively, we have

$$\varepsilon_1 = \alpha \Delta t\,, \quad \varepsilon_2 = -\frac{\sigma}{E}\,.$$

Assuming that the gate stays rigid until it breaks, the length of the ram needs to stay constant during the heating process. This is facilitated by increasing the compressive stress inside the ram. The condition of keeping the length of the ram constant then translates to

$$\varepsilon_1 + \varepsilon_2 = 0 \quad \Rightarrow \quad \Delta t = \frac{\sigma}{\alpha E}\,.$$

The increase in temperature is therefore directly proportional to an increase in compressive stress. Thus, since we know that the gate yields when $\sigma = \sigma_{\max}$, the required change of temperature is

$$\Delta t = \frac{\sigma_{\max}}{\alpha E} \doteq 158\,\mathrm{K}\,.$$

In order to break through the gate, the orcs need to increase the temperature of the ram by $\Delta t = 158\,\mathrm{K}$.

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Problem FoL.16 ... refracter

A ray of white light is incident on a thin glass wall of a hollow prism filled with carbon disulphide. The angle of incidence (measured from the normal to the interface) is $\varepsilon_1 = 50^\circ$. The walls of the prism are made from thin plane-parallel plates. The apex angle of the prism is $\varphi = 60^\circ$. Compute the angular width of the light exiting the prism (in degrees). The refractive index of the glass is $n_{\mathrm{cs}} = 1.518$ for red light and $n_{\mathrm{fs}} = 1.599$ for violet light; the refractive index of carbon disulphide for red light is $n_{\mathrm{c}} = 1.618$ and for violet light $n_{\mathrm{f}} = 1.699$. Assume that the surrounding air has refractive index $n = 1$ for all wavelengths.

Faleš struggled with old physics exercise books.

We can compute the desired angle α as the difference between the angles of refraction of red light and violet light. Each ray of light going through the prism refracts four times – twice while entering the prism and twice while exiting it. However, we can simplify the situation when we come to realize that the pairs of refractions can be considered as only one refraction directly from the air to the carbon disulphide because the plates are plane-parallel. We could show this by repeatedly using Snell's law.

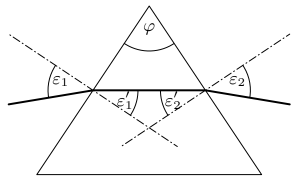

Fig. 1: Geometry of the prism.

Let us denote the final angle of refraction for a particular wavelength δ. The geometry tells us

$$\delta = \varepsilon_1 + \varepsilon_2 - \varphi \,,$$

where ε_2 is the angle of incidence for the second wall (the one through which the light escapes the prism). The angle α can be expressed as

$$\alpha = \delta_{\mathrm{f}} - \delta_{\mathrm{c}} \,.$$

The indices f and c denote violet and red light, respectively. The angles ε_1 and φ are the same for all wavelengths and therefore, they cancel out in the formula. We have

$$\alpha = \varepsilon_{2\mathrm{f}} - \varepsilon_{2\mathrm{c}} \,.$$

We find this angle as follows

$$\sin \varepsilon_{2i} = \frac{\sin \varepsilon_2' n_i}{n} \,,$$

where the primed angles are those inside the prism and the index i is f or c, depending on the wavelength. For ε_2', it follows that

$$\varepsilon_{2i}' = \varphi - \varepsilon_{1i}' \,,$$

$$\sin \varepsilon_{1i}' = \frac{\sin \varepsilon_{1i} n}{n_i} \,.$$

Finally,

$$\alpha = \arcsin\left(\frac{\sin\left(\varphi - \arcsin \frac{\sin \varepsilon_1 n}{n_{\mathrm{f}}}\right) n_{\mathrm{f}}}{n}\right) - \arcsin\left(\frac{\sin\left(\varphi - \arcsin \frac{\sin \varepsilon_1 n}{n_{\mathrm{c}}}\right) n_{\mathrm{c}}}{n}\right) .$$

After numerical computations, we have $\alpha \doteq 10.1^\circ$.

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Problem FoL.17 ... the Long Night

What amount of work must be done to extend a day by 4 hours? Consider the Earth to be a homogeneous solid sphere with radius $R = 6\,378\,\mathrm{km}$ and mass $M = 5.972 \cdot 10^{24}\,\mathrm{kg}$.

Kuba would like to have more time to sleep.

In order to extend the day on Earth, we must slow down its rotation. In other words, we're asking by how much the kinetic energy of the Earth's rotation needs to be changed. We know that rotational kinetic energy is given by the relation $E_\mathrm{r} = \frac{1}{2}J\omega^2$, where J is the moment of inertia of the Earth and ω is the angular velocity of its rotation, which can be calculated from the rotation period (i.e. length of the day). The moment of inertia of a solid sphere is $J = \frac{2}{5}MR^2$. Then,

$$W = \frac{1}{2}J\left(\left(\frac{2\pi}{T_1}\right)^2 - \left(\frac{2\pi}{T_2}\right)^2\right) = \frac{1}{5}MR^2\left(\left(\frac{2\pi}{T_1}\right)^2 - \left(\frac{2\pi}{T_2}\right)^2\right),$$

where T_1 is the original length and T_2 is the new length of the day. Numerical evaluation gives us $W = 6.817 \cdot 10^{28}\,\mathrm{J}$.

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Problem FoL.18 ... thick glass

Lukáš wants to admire the Black rock from his dormitory window, but another building is standing in the way. Lukáš is $d = 400\,\mathrm{m}$ away from the building and looks (approximately) right at it, so the building appears as a rectangle with height $h = 70\,\mathrm{m}$ and width $w = 50\,\mathrm{m}$. Determine the solid angle (in steradians) covered by the building.

Mirek realized there's landmark next to the dormitory.

Notice that the distance d is about one order of magnitude bigger than h, w. Then, the solid angle Ω can be sufficiently approximated by the ratio of the visible surface area of the building and a sphere of radius d, multiplied by the full solid angle 4π. Written as a formula,

$$\Omega = 4\pi\frac{wh}{4\pi d^2} = \frac{wh}{d^2} \doteq 0.021\,9\,.$$

The exact result can be obtained by integration

$$\int \frac{\mathrm{d}S}{r^2}\cos\alpha\,,$$

where $\mathrm{d}S$ is an area element, r its distance from the observer and α is the angle between the normal $\boldsymbol{n}$ of the area element and the vector $\boldsymbol{r}$ from the observer to it. Computing this in Cartesian coordinates,

$$\int_{-w}^{w}\int_{-h}^{h} \frac{d}{\left(d^2+x^2+y^2\right)^{3/2}}\,\mathrm{d}x\,\mathrm{d}y = 4\,\mathrm{arctg}\,\frac{\frac{wh}{2d}}{\sqrt{4d^2+w^2+h^2}} \doteq 0.022\,.$$

We can see that the approximation was accurate enough.

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Problem FoL.19 ... wrong slope

Construction workers don't always follow the blueprints precisely. After one "hard night", they read the plans incorrectly and built a road curve tilted to the other side than it is usually done. By how many kilometers per hour did the maximum speed, with which a car may cross the curve without slipping, decrease? The radius of the curve is $r = 100\,\mathrm{m}$, the angle by which it's tilted is $\alpha = 5^\circ$ and the static friction coefficient between a tyre and the road surface is $f = 0.55$. The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. *Pikoš was watching construction workers.*

Consider the problem in the reference frame of the car. There are four forces acting on the car when it's crossing the curve:

- *force of gravity* $\boldsymbol{F}_g = m\boldsymbol{g}$, where m is the mass of the car and $\boldsymbol{g}$ the acceleration due to gravity,
- *centrifugal force* $\boldsymbol{F}_\mathrm{o} = mv^2\hat{r}/r$, where v is the speed of the car, r is the radius of the curve and $\hat{r}$ is the horizontal unit vector pointing away from the center of curvature of the curve,
- *reaction force from the road* $\boldsymbol{F}_\perp = F_\perp(-\sin\alpha, \cos\alpha)$, which is normal to the road, and
- *friction force* $\boldsymbol{F}_\mathrm{t} = F_\mathrm{t}(-\cos\alpha, -\sin\alpha)$, which is tangent to the road and whose magnitude mustn't exceed $fF_\perp$, where f is the static friction coefficient. Provided that the centrifugal force is large enough, the friction force will act against it. Since we're looking for the maximum speed, which is limited just by the maximum magnitude of this force, let's suppose $F_\mathrm{t} = fF_\perp$.

The total force is $\boldsymbol{F} = \boldsymbol{F}_g + \boldsymbol{F}_\mathrm{o} + \boldsymbol{F}_\perp + \boldsymbol{F}_\mathrm{t}$. Since the car doesn't move in its own reference frame, we must have $\boldsymbol{F} = \boldsymbol{0}$, so

$$m(0, -g) + \frac{mv^2}{r}(1, 0) + F_\perp(-\sin\alpha, \cos\alpha) + fF_\perp(-\cos\alpha, -\sin\alpha) = \boldsymbol{0}\,.$$

We can express $F_\perp$ from both components and set those expressions equal; from that equality, we can express the speed

$$v = \sqrt{gr\frac{\sin\alpha + f\cos\alpha}{\cos\alpha - f\sin\alpha}}\,.$$

For $\alpha = 5^\circ$, we find out that the maximum speed is approximately $92.3\,\mathrm{km{\cdot}h^{-1}}$; for $\alpha = -5^\circ$, it's $74.9\,\mathrm{km{\cdot}h^{-1}}$, so the maximum speed decreased by $17.4\,\mathrm{km{\cdot}h^{-1}}$.

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Problem FoL.20 ... minimal force

Consider three particles with positive charges $Q_1 = 1\,\mathrm{C}$, $Q_2 = 2\,\mathrm{C}$ and $Q_3 = 4\,\mathrm{C}$ placed in vacuum. Find the minimal possible magnitude of the force between Q_1 and Q_3, if you know that the force between Q_1 and Q_2 is $F_{12} = 1\,\mathrm{N}$ and the force between Q_2 and Q_3 is $F_{23} = 4\,\mathrm{N}$.
Náry was trying to minimise his repulsiveness.

We use Coulomb's law

$$F_{AB} = \frac{1}{4\pi\varepsilon}\frac{Q_A Q_B}{d^2}\,,$$

where d is the distance between two charges Q_A, Q_B and ε is the permittivity of the surrounding material. Using this formula, we start by calculating the distances d_{12} and d_{23}

$$d_{12} = \sqrt{\frac{1}{4\pi\varepsilon}\frac{Q_1Q_2}{F_{12}}}\,,$$
$$d_{23} = \sqrt{\frac{1}{4\pi\varepsilon}\frac{Q_2Q_3}{F_{23}}}\,.$$

Because the repulsive force between particles decreases with their distance, we get the minimal repulsive force for the maximal distance between charges Q_1 and Q_3, which is $d_{13} = d_{12} + d_{23}$. Coulomb's law then gives us

$$F_{13} = \frac{1}{4\pi\varepsilon}\frac{Q_1Q_3}{\left(\sqrt{\frac{1}{4\pi\varepsilon}\frac{Q_1Q_2}{F_{12}}}+\sqrt{\frac{1}{4\pi\varepsilon}\frac{Q_2Q_3}{F_{23}}}\right)^2}\,,$$
$$F_{13} = \frac{Q_1Q_3}{\left(\sqrt{\frac{Q_1Q_2}{F_{12}}}+\sqrt{\frac{Q_2Q_3}{F_{23}}}\right)^2}\,.$$

Plugging in the numbers, we get the value 0.50 N.

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Problem FoL.21 ... laser pointer

A monochromatic laser points perpendicularly at a solid screen. We mark the point on the screen at which the laser is pointing when there is nothing but air in between. Now, we place a transparent plate made from an optically denser material with thickness $d = 2\,\mathrm{cm}$ between the laser and the screen. The laser beam enters the plate at an angle $\alpha = 30^\circ$ from the normal and emerges on the opposite side of the plate. The laser is now pointing at a point which is shifted from the original one by $\delta = 0.6\,\mathrm{cm}$. What is the refractive index n_2 of the plate? The refractive index of air is $n_1 = 1$. *Meggy tried geometry.*

We'll project our scene onto one plane, in which the laser beam propagates. Both before entering and after exiting the plate, the laser beam points in the same direction, which means that the shift had to occur within the plate. According to Snell's law, the relation

$$\frac{n_1}{n_2} = \frac{\sin\beta}{\sin\alpha}$$

holds, where β is the angle of refraction within the plate. Imagine the propagation of the beam without the plate. Let us call this beam p and the real refracted beam q. Both rays enter the plate at one point C, the beam p emerges from the plate at point P and the beam q at another point Q. The distance δ is the length of the perpendicular from point Q to the beam p; we denote its foot by R and the point opposite to C by D (then, $|\mathsf{CD}| = d$). The triangles PQR and PCD are similar.

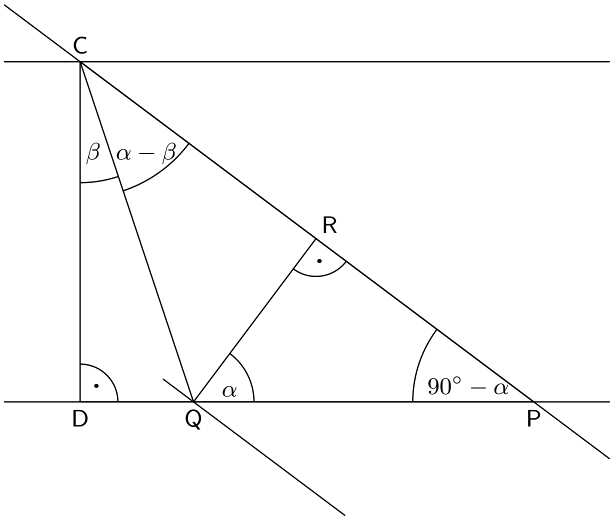

Fig. 2: Geometry of refraction of the laser.

We can see that

$$|\mathsf{PQ}| = \frac{\delta}{\cos\alpha} \doteq 0.692\,8\,\mathrm{cm}\,,$$
$$|\mathsf{PD}| = \frac{d}{\operatorname{tg}\left(\frac{\pi}{2}-\alpha\right)} = d\operatorname{tg}\alpha \doteq 1.154\,7\,\mathrm{cm}\,,$$
$$\operatorname{tg}\beta = \frac{|\mathsf{QD}|}{d} = \frac{|\mathsf{PD}|-|\mathsf{PQ}|}{d} \doteq 0.231\,.$$

From that, we get

$$\beta \doteq 13.0^\circ\,.$$

Thus, we can express

$$n_2 = \frac{n_1\sin\alpha}{\sin\beta} \doteq 2.22\,.$$

The refractive index of the screen is 2.22.

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Problem FoL.22 ... when to stop a catapult

Consider a massless rod in Earth's homogeneous gravity field. The rod rotates about an axis that is perpendicular both to the gravitational acceleration and to the rod. At a distance $R = 1\,\mathrm{m}$ from the axis, we place a weight $M = 10\,\mathrm{kg}$, and on the other side, we place a projectile with mass $m = 500\,\mathrm{g}$ at a distance $r = 10\,\mathrm{m}$ from the axis. Afterwards, we set the catapult (rod) to horizontal position. When the catapult is ready, we release the rod and let it rotate until it reaches an angle of φ with the horizontal, where it's stopped abruptly. Find the angle φ

that maximizes the range of the catapult. To simplify the calculations, we assume that the projectile hits the ground at the same height as the height at which it was released when the rod was stopped. Also, the projectile cannot slide on the rod.

Lubošek wanted to shoot at his roommate.

First, let us show on which variables the range d depends. Elementary ballistics states that

$$d = v_x t = \frac{2}{g} v_y v_x = \frac{2}{g} v^2 \sin\varphi \cos\varphi\,,$$

where v is the initial velocity of the projectile, v_x and v_y are its components and g is the gravitational acceleration. At the moment of release of the projectile, we have

$$d = \frac{2}{g} r^2 \omega^2 \sin\varphi \cos\varphi\,,$$

where ω denotes the angular velocity of the rod right before the release.

In the next step, we use the law of conservation of mechanical energy before and after release

$$g(MR - mr)\sin\varphi = \frac{1}{2}(MR^2 + mr^2)\omega^2\,.$$

Substituting into the equation for the range, we obtain

$$d = \frac{4r^2(MR - mr)}{MR^2 + mr^2} \sin^2\varphi \cos\varphi\,.$$

It is worth noticing that the range does not depend on gravitational acceleration and all the mechanical parameters of the catapult stand only for a multiplicative factor.

In the last step, we use the fact that the range is zero for $\varphi = 0^\circ$ and $\varphi = 90^\circ$ and positive for any value in this interval. From that, it follows that if the function $d(\varphi)$ has only one stationary point (point with zero derivative), then it'll also be the maximum of that function. The condition

$$\frac{\mathrm{d}d}{\mathrm{d}\varphi} = 0$$

leads us to the equation

$$2\cos^2\varphi - \sin^2\varphi = 0\,.$$

Finally, through trigonometric identities, we get $\varphi = \arccos\left(1/\sqrt{3}\right) \doteq 54.74^\circ$.

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Problem FoL.23 ... alternating-direct 1

Consider the circuit in the figure 3, the DC source has voltage $U_\mathrm{j} = 4.5\,\mathrm{V}$*, the AC source has voltage amplitude* $U_\mathrm{s} = 5\,\mathrm{V}$ *and frequency* $f = 50\,\mathrm{Hz}$*. The resistor's resistance is* $R = 100\,\mathrm{k\Omega}$ *and the capacitor's capacity is* $C = 10\,\mathrm{nF}$*. What's the expected charge on the capacitor, in* nC*?*

Xellos doesn't like broccoli, that's why he made a problem without it.

The two sources are a trap – all elements of the circuit are linear, so we can view it as a superposition (sum) of two circuits – one with the DC and the other with the AC source. The charge on the capacitor is a superposition of the charges in both cases; however, the expected

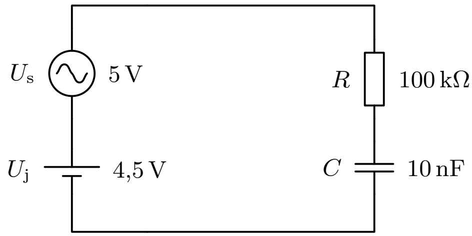

Fig. 3: Circuit diagram.

value with the AC source is 0, so the result is the same as if we only had the DC source. In that case, there's no current passing through the resistor, so we can discard it and the result is $Q = CU_\mathrm{j} = 45\,\mathrm{nC}$.

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Problem FoL.24 ... tick tock

There are two clocks in a box. One is a balance wheel clock (like a usual watch; alternatively, a digital clock) and the other a pendulum clock – its pendulum consists of a rod (length $l = 30\,\mathrm{cm}$ and mass $m = 300\,\mathrm{g}$) with a disk at its lower end (diameter $d = 10\,\mathrm{cm}$ and area density $\sigma = 7.5 \cdot 10^{-2}\,\mathrm{kg{\cdot}m^{-2}}$). The disk is attached lengthwise in the only direction in which the pendulum can move (oscillate). A small rocket lifts the box with acceleration $a = 5g$ (g is the gravitational acceleration of the Earth) up to the height $h = 30\,\mathrm{km}$, where it lets the box move with unchanged inertia in Earth's gravity. Assume that the gravitational field is homogeneous with $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$ and neglect the drag force. By how much will the times on the two clocks differ (in absolute value) at the highest point of the box's trajectory?

Fales remade the relativistic paradox.

We need to realize that unlike a clock with a balance wheel (digital clock), the pendulum clock depends on the gravitational acceleration. During the ascent of the rocket, it is effectively six times greater $(1 + 5)$ than the acceleration due to Earth's gravity. On the contrary, after being released, the box will be in a weightless state – it will be in free fall (with initial velocity v pointing upward).

The digital clock remains unaffected, which means that the time it shows is the sum $t = t_1 + t_2$, where t_1 is the time during which the box was being lifted and t_2 is the time during which it was in a weightless state since being released till reaching the highest point. They can be obtained as

$$t_1 = \sqrt{\frac{2h}{a}}\,,$$

$$t_2 = \frac{v}{g} = \frac{t_1 a}{g} = \frac{\sqrt{2ha}}{g}\,.$$

The period of the pendulum clock is computed as

$$T = 2\pi\sqrt{\frac{I+ml^2}{mgl}} = 2\pi\sqrt{\frac{l+\frac{I}{ml}}{g}} = 2\pi\sqrt{\frac{l_{\mathrm{red}}}{g}}\,,$$

where I is the moment of inertia of the system of the rod and disc about the axis crossing its center of mass, whose distance from the actual axis of rotation is l. Using the Parallel axis theorem, we get the moment of inertia about the rotational axis as $I + ml^2$. However, the only important thing is that the original expression can be transformed to the same form as the one for a mathematical pendulum with some new length l_{red}, the numerical value of which isn't important for us, since we found out that the period is proportional to $g^{-\frac{1}{2}}$. When the acceleration gets six times greater, the time of ascent measured by the pendulum clock will be $t_1^{\mathrm{k}} = \sqrt{6}t_1$ (the period will be $\sqrt{6}$-times smaller, so the measured time will be bigger). The clock stops in the weightless state, so the time measured by it in this state will be $t_2^{\mathrm{k}} = 0\,\mathrm{s}$.

The desired time difference τ is

$$\tau = \left|\left(1-\sqrt{6}\right)\sqrt{\frac{2h}{a}} + \frac{\sqrt{2ha}}{g}\right|\,.$$

After numerical evaluation, we get $\tau \doteq 124\,\mathrm{s}$.

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Problem FoL.25 ... sleepyhead

Exactly at 6:30 in the morning, a solution of a radiopharmaceutical containing radioactive ^{18}F *with volumetric activity* $330\,\mathrm{MBq\cdot ml^{-1}}$ *was prepared for a patient.* $1\,\mathrm{ml}$ *of this solution should have been applied at exactly 8:00 on the same day. However, the patient was late and he took his shot at 8:40. What volume of the solution of the radiopharmaceutical (in ml) should the patient get so that the dose would be equivalent to the originally planned one? The half-life of* ^{18}F *is* $109\,\mathrm{min}$ *and its decay product is stable oxygen.*

Wake up early and don't miss your deadlines.

The activity of the radiopharmaceutical at 8:00 will be

$$A(t) = A_0 \exp\left(-\frac{\ln 2}{T}t\right)\,,$$

where A_0 is the original activity, T is the half-life and $t = 90\,\mathrm{min}$ is the time since preparation. Similarly, we can calculate the activity of the solution at 8:40, if we just substitute a different time $t' = 130\,\mathrm{min}$. If we know that at 8:00, one milliliter of the solution would be enough, then with the current activity at 8:40, we will need a proportionally larger volume

$$\frac{V'}{V} = \left(A_0 \mathrm{e}^{-\frac{\ln 2}{T}t}\right) / \left(A_0 \mathrm{e}^{-\frac{\ln 2}{T}t'}\right) = \exp\left(\frac{\ln 2\,(t'-t)}{T}\right) = 2^{\frac{t'-t}{T}} = 1.29\,,$$

and therefore, the needed volume will be $V' = 1.29\,\mathrm{ml}$ of the solution.

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Problem FoL.26 ... Martian

What is the maximal pressure (in μPa*) that a spherical balloon the size of Mars can hold, if it's made from monolayer graphene? The radius of Mars is* $R_\mathrm{M} = 3\,390\,\mathrm{km}$*, the maximal tension in one layer of graphene is* $\sigma_\mathrm{max} = 42\,\mathrm{N\cdot m^{-1}}$*. Neglect the gravitation of the gas.* *Elon Musk.*

Divide the balloon (with gas) into two equal hemispheres. Each one is pushed away from their (fictitious) contact circle by the pressure of the gas and they're held together by the tension on the circumference of this circle. The maximal force from this tension is $2\pi R_\mathrm{M}\sigma_\mathrm{max}$. This force has to be $\geq$ than the force exerted by pressure on their contact circle, which is equal to $\pi R_\mathrm{M}^2 p$. Expressing the pressure and plugging in the numbers, we get

$$p_\mathrm{max} = \frac{2\sigma_\mathrm{max}}{R_\mathrm{M}} \doteq 25\,\mathrm{\mu Pa}\,.$$

The maximal pressure that can be held in the balloon is $p_\mathrm{max} \doteq 25\,\mathrm{\mu Pa}$.

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Problem FoL.27 ... nitrogen suffocates the audience

What volume V *would the nitrogen from a* $V_\mathrm{D} = 7.0\,\mathrm{l}$ *Dewar flask take after boiling away in standard conditions, i.e.* $t = 20\,\mathrm{^\circ C}$ *and* $p_\mathrm{a} = 1\,013\,\mathrm{hPa}$*? We are interested only in the volume of the evaporated nitrogen, so you can also consider an equivalent problem of determining the volume of a room which contained only vacuum and liquid nitrogen at the beginning and only* N_2 *gas at standard temperature and pressure after the nitrogen evaporated. The density of liquid nitrogen is* $\varrho_\mathrm{L} = 808\,\mathrm{kg\cdot m^{-3}}$*, the molar mass of nitrogen is* $M_\mathrm{m} = 28.0\,\mathrm{g\cdot mol^{-1}}$*, the molar gas constant is* $R = 8.31\,\mathrm{J\cdot K^{-1}\cdot mol^{-1}}$ *and the density of nitrogen gas under standard conditions is* $\varrho_\mathrm{G} = 1.16\,\mathrm{kg\cdot m^{-3}}$*.* *Karel was wondering what would have to happen to kill the audience of a liquid nitrogen experiment.*

The simplest solution is to use the fact that the mass of nitrogen m is the same before and after the evaporation. We have, "using the rule of three"

$$m = \varrho_\mathrm{L} V_\mathrm{D} = \varrho_\mathrm{G} V \quad \Rightarrow \quad V = V_\mathrm{D}\frac{\varrho_\mathrm{L}}{\varrho_\mathrm{G}} \doteq 4.9\,\mathrm{m^3}\,.$$

We got our solution very quickly: the nitrogen would, after evaporation, take the volume $4.9\,\mathrm{m^3}$.

However, the problem was set in a little bit tricky way in order to make you want to use a slower path, which also leads to the right solution. Let's also show that solution. We start from the equation of state for an ideal gas, because we know the pressure and the temperature of the final state:

$$p_\mathrm{a} V = nRT\,,$$

where p_a and R are given, V is to be determined and $T \doteq 293\,\mathrm{K}$ can be expressed from $t = 20\,\mathrm{^\circ C}$. The amount of nitrogen n can then be expressed as

$$n = \frac{m}{M_\mathrm{m}} = \frac{\varrho_\mathrm{L} V_\mathrm{D}}{M_\mathrm{m}}\,.$$

We get an expression for the volume as

$$p_\mathrm{a}V = \frac{\varrho_\mathrm{L}V_\mathrm{D}}{M_\mathrm{m}}RT \quad \Rightarrow \quad V = \frac{\varrho_\mathrm{L}V_\mathrm{D}}{p_\mathrm{a}M_\mathrm{m}}RT \doteq 4.9\,\mathrm{m}^3\,.$$

Thus, within the required precision, we obtain the same result. There is a difference in the next significant digits, but it's caused by rounding and doesn't have physical meaning.

If any of you are wondering whether experiments with liquid nitrogen would suffocate the audience, it would be necessary to reduce the oxygen content in the room at least under 11 % (and probably under 8 %). As rooms are usually ventillated, it is not easy to suffocate someone.

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Problem FoL.28 ... exoplanet

The exoplanet Kepler-138c has radius $R_\mathrm{K} = 1.2\,R_\mathrm{E}$, where R_E is the radius of the Earth, and density equal to that of the Earth. The inhabitants dug a tunnel straight through the center of the planet (along the diameter). How much longer will it take to free-fall through this tunnel (without friction or propulsion) than through a similar tunnel on Earth? Give your answer in seconds. Assume that both planets are homogeneous. *Filip was digging in his garden.*

From Gauss's law, we know that the mass "above" the falling object has no influence on it. This means that the force acting on an object at a distance r from the center is

$$m\ddot{r} = -\frac{4\pi r^3 \varrho G m}{3r^2}\,,$$
$$\ddot{r} = -\frac{4}{3}\pi\varrho G r\,.$$

From this, we can see that the period of the oscillations depends only on the density, so the difference in times is zero.

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Problem FoL.29 ... a hideout on the Moon

Consider two people (let's call them Iva and Radek) standing on the surface of the Moon. Find their smallest possible separation needed so that a voyeur (let's call him Aleš), standing on the summit of Mt. Palomar in the distance $r = 3.8\cdot 10^5\,\mathrm{km}$ from the pair, would be able to resolve them from one another using a telescope with aperture diameter $5\,\mathrm{m}$. Take Iva and Radek to be two point-like sources emitting light with wavelength $500\,\mathrm{nm}$. *Dominika making up gossips.*

Light is known to be diffracted by individual components of the optical system we happen to be using – lenses, apertures etc. Thus the image of a point-like source consists of multiple diffraction rings. The Rayleigh criterion then says that two point-like sources can be distinguished when the first diffraction minimum of the image of one source coincides with the maximum of another. Expressing this mathematically for the given telescope, we get

$$\alpha = \frac{1.22\lambda}{d}\,,$$

where λ is the wavelength, d is the aperture diameter and the factor of 1.22 follows from the intensity profile of the diffraction pattern which can be expanded into Bessel's function of the first kind. Denoting the separation of Iva and Radek by x, we clearly have $x \ll r$, so the corresponding small angle approximation $\operatorname{tg}\alpha \approx \sin\alpha \approx \alpha$ may be invoked. We then have

$$\alpha \approx \frac{x}{r}\,.$$

and so

$$x = 1.22\frac{\lambda r}{d} \doteq 46.4\,\mathrm{m}\,.$$

It is much to Aleš's displeasure that he will be able to distinguish the two only after they move away to a distance $x = 46.4\,\mathrm{m}$ from each other.

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Problem FoL.30 ... melted down thermistor

A thermistor is a semiconductor component, the resistance of which decreases significantly with temperature according to the formula $R = R_0 \exp\left(c\left(1/T - 1/T_0\right)\right)$, where the temperatures T, T_0 are substituted in K, $R_0 = 120\,\Omega$ is its resistance at room temperature $T_0 = 25\,^\circ\mathrm{C}$ and $c = 3.0\cdot 10^3\,\mathrm{K}$. This formula loses validity at the temperatur $T_\mathrm{t} = 150\,^\circ\mathrm{C}$, when the thermistor starts to melt down. The power output of heat transfer from the thermistor at a temperature T to its surroundings at the temperature T_0 is $P_\mathrm{chl} = k(T - T_0)$, where $k = 4.5\cdot 10^{-4}\,\mathrm{W{\cdot}K^{-1}}$. Determine the highest voltage to which the thermistor can be connected without being heated up to the temperature T_t or higher (after a sufficiently long time).

Xellos remembered his laboratory practice.

We want to find the maximal voltage at which the thermistor reaches thermal equilibrium – there exists a temperature at which all the Joule heat is being transferred to the surroundings. Then, it holds true that

$$k(T - T_0) = \frac{U^2}{R}\,,$$

$$U = \sqrt{k(T - T_0)R_0 \mathrm{e}^{c\left(\frac{1}{T} - \frac{1}{T_0}\right)}}\,.$$

The expression under the square root grows linearly with T in the limit of infinite temperatures. However, on the scale of thousands of kelvins (much more than T_t), it has only one local maximum. Therefore, we can search for a point with zero first derivative:

$$\mathrm{e}^{c\left(\frac{1}{T} - \frac{1}{T_0}\right)} + (T - T_0)\mathrm{e}^{c\left(\frac{1}{T} - \frac{1}{T_0}\right)} c\left(-\frac{1}{T^2}\right) = 0\,,$$

$$T^2 = c(T - T_0)\,,$$

$$T = \frac{c \pm \sqrt{c^2 - 4cT_0}}{2} \doteq 336\,\mathrm{K}$$

for the minus sign (the plus sign leads to a minimum at a temperature around $2\,700\,\mathrm{K}$). The voltage at which the termistor has this equilibrium temperature is $U = 0.81\,\mathrm{V}$.

Another option is to read out the maximum from a graph of $U(T)$.

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Problem FoL.31 ... rotation rotation

Verča had a fancy party hat on, but she moved too fast and it fell to the ground and began to roll. The conical hat has a slant height of $R = 30\,\mathrm{cm}$, the radius of the base circle is $r = 10\,\mathrm{cm}$ and during its movement, the cone rotates about its stationary apex with an angular velocity ω_y. The cone also rotates about its rotational axis with an angular velocity $\omega_o = 5.0\,\mathrm{rad\cdot s^{-1}}$. What is the angular velocity of any point on the surface of the cone with respect to the immediate axis of rotation? The cone rolls on a rough surface. *Mirek combined rotations.*

The angular velocity ω, which we are supposed to find, is given by vector addition of the angular velocities ω_o and ω_y. The vectors are shown in the figure 4.

Since the movement takes place on a rough surface, the cone cannot slip, so it moves circularly about its apex. We can immediately see the simple relation between ω_y and ω_o

$$R\omega_y = r\omega_o\,. \tag{2}$$

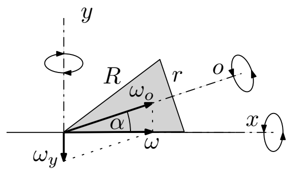

Fig. 4: Analysis of the rotational motion.

We introduce the angle α between ω and ω_o; using the cosine law, we can write

$$\omega_y^2 = \omega^2 + \omega_o^2 - 2\omega\omega_o\cos\alpha\,. \tag{3}$$

The angle α is the angle at the apex and satisfies the relation

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{r^2}{R^2}} = \frac{\omega}{\omega_o}\,.$$

Substituting into the cosine law (3), we obtain

$$\omega^2 = \omega_o^2 - \omega_y^2\,;$$

that is the Pythagorean theorem, thus ω must lie in the ground plane, as the picture hints. Using the relation (2), we can finally express the magnitude of the angular velocity ω

$$\omega = \omega_o\sqrt{1-\frac{r^2}{R^2}} = 4.7\,\mathrm{s^{-1}}\,.$$

The angular velocity is $\omega = 4.7\,\mathrm{s^{-1}}$.

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Problem FoL.32 ... superpowers

If Aleš were to pick a superpower, he would choose the ability to see the intensities of electric and magnetic field. Let's assume that he has such an ability. Imagine that he stands $1.8\,\mathrm{m}$ *away from a point source of light with power* $P = 250\,\mathrm{W}$. *What is the effective value of the magnetic field (magnetic induction) which he feels?*

Dominika thinks of the consequences of the organizers' wishes.

In this case, the energy of electromagnetic waves is conserved – if we construct a sphere with radius r centered at the light source, all the energy from the source must pass through its surface. The energy that passes through that surface in a unit of time must be the same as the energy produced by the source over the same time, i.e. the power P. The intensity I at the surface of the sphere is

$$I = \frac{1}{c\mu_0} E_{\mathrm{ef}}^2 = c\varepsilon_0 E_{\mathrm{ef}}^2 \,,$$

where c is the speed of light, μ_0 is vacuum permeability and ε_0 vacuum permittivity; the relation between those constants is $c^2\varepsilon_0\mu_0 = 1$. Next, we use the relation between magnetic induction B_{ef} and electric intensity E_{ef}: $E_{\mathrm{ef}} = cB_{\mathrm{ef}}$ (thus, I is the density of EM energy multiplied by the speed of the wave). Using these relations, we isolate

$$B_{\mathrm{ef}} = \sqrt{\frac{P\mu_0}{4\pi r^2 c}} = 1.6 \cdot 10^{-7}\,\mathrm{T} \,.$$

Aleš feels a magnetic field with induction $1.6 \cdot 10^{-7}\,\mathrm{T}$.

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Problem FoL.33 ... ball-ey

A small homogeneous spherical ball with radius $r = 3\,\mathrm{cm}$ *is placed in a fixed spherical hole with radius* $R = 10\,\mathrm{cm}$. *The ball can roll without slipping. We move the ball from its equilibrium position a little bit. What will be the period of its oscillations?*

Guljočka v jamočke, Tom kept repeating.

Let's forget about the ball not rolling on a flat surface. If the centre of the ball moves with a speed v, then the ball will rotate with the angular speed $\omega = v/R$. Then, its kinetic energy will be

$$E_{\mathrm{k}} = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 \,,$$

where m is the mass and $I = \frac{2}{5} m r^2$ is the moment of inertia of the ball. After some simplifications, we get

$$E_{\mathrm{k}} = \frac{1}{2} m v^2 + \frac{1}{5} m r^2 \omega^2 = \frac{7}{10} m v^2 \,.$$

If the ball is displaced by an angle φ (measured with respect to the centre of the hole; in the equilibrium position, $\varphi = 0$), we may express the speed of its centre as $v = (R - r)\mathit{\Omega}$, where $\mathit{\Omega} = \mathrm{d}\varphi/\mathrm{d}t$ is the angular speed of the centre of the ball with respect to the centre of the hole. The potential energy of the ball (with zero in the equilibrium position) is

$$E_{\mathrm{p}} = mg(R - r)\left(1 - \cos(\varphi)\right) \,.$$

Since φ is very small, we can approximate it as $\cos(\varphi) \approx 1 - \varphi^2/2$. Then, the law of energy conservation is as follows:

$$\frac{7}{10} m (R-r)^2 \Omega^2 + mg(R-r)\frac{\varphi^2}{2} = \text{const}\,.$$

It describes a harmonic oscillator, whose solution is

$$\varphi(t) = \varphi_0 \sin\left(\frac{2\pi}{T} t\right)\,,$$

where φ_0 is the amplitude and T is the period

$$T = 2\pi\sqrt{\frac{7(R-r)}{5g}} \doteq 0.63\,\mathrm{s}\,.$$

The ball oscillates with period $T \doteq 0.63\,\mathrm{s}$.

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Problem FoL.34 ... diodes

Consider two diodes satisfying the ideal Shockley diode law

$$I(U) = I_\mathrm{S}\left(\exp\left(\frac{U}{V_\mathrm{T}}\right) - 1\right)\,,$$

connected in series with a DC source with voltage $V = 1\,\mathrm{V}$ and internal resistance $100\,\Omega$. The saturation current of one diode is $I_\mathrm{S} = 1\cdot 10^{-11}\,\mathrm{A}$, the saturation current of the other one is twice as large. The thermal voltage is $V_\mathrm{T} = 26\,\mathrm{mV}$. What is the total electric power (in mW) produced on the diodes?

Hint *Since we only want a numerical result, it's sufficient to solve the equations numerically.*

Janči was thinking about diodes in series.

Denote the potentials as in the figure; the zero potential will be on the negative pole of the source. Using Ohm's and Shockley's law, we write

$$V - \varphi_2 = RI\,,$$
$$I_{\mathrm{S}2}\left(\mathrm{e}^{\frac{\varphi_2-\varphi_1}{V_\mathrm{T}}} - 1\right) = I\,,$$
$$I_{\mathrm{S}1}\left(\mathrm{e}^{\frac{\varphi_1}{V_\mathrm{T}}} - 1\right) = I\,.$$

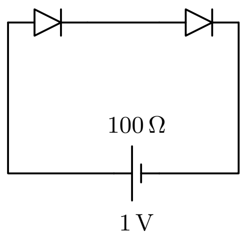

Fig. 5: Circuit diagram.

Then, we express $\varphi_2 = V - RI$ and $\exp(\varphi_1/V_\mathrm{T}) = I/I_{\mathrm{S}1} + 1$ and plug into the second equation

$$I_{\mathrm{S}2}\left(\mathrm{e}^{\frac{V-IR}{V_\mathrm{T}}}\left(\frac{I}{I_{\mathrm{S}1}} + 1\right)^{-1} - 1\right) = I\,.$$

Notice that the currents I_{S1} and I_{S2} have to be very small, compared to the current I: if it weren't the case, the voltages on the diodes would be comparable to V_{T} and the voltage on the resistor would then have to be similar to V, almost $1\,\mathrm{V}$. That would, however, mean that the current I is about V/R, which is a contradiction.

This allows us to neglect the small currents with respect to I and, after simplifying a bit, we get

$$I = \sqrt{I_{\mathrm{S1}} I_{\mathrm{S2}}}\,\mathrm{e}^{\frac{V-RI}{2V_{\mathrm{T}}}}\ .$$

This equation can be solved numerically for the current. The result is

$$I \doteq 0.75\,\mathrm{mA}\,,$$

where we see that the error due to our approximation is about $I_{\mathrm{S}}/I \doteq 10^{-8}$.

Because the current I is the same for both diodes and the total voltage drop across them is φ_2, the total power is

$$P = \varphi_2 I = (V - RI)I \doteq 0.69\,\mathrm{mW}\,.$$

The power produced on the diodes is $P = 0.69\,\mathrm{mW}$.

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Problem FoL.35 ... gardener's problem

An amateur gardener is watering his garden. He took a cylindrical bucket with base area $S = 420\,\mathrm{cm}^2$ and filled it with water up to a height $h = 40\,\mathrm{cm}$. Afterwards, he made a hole with area $s = 2\,\mathrm{cm}^2$ in the bottom base and started watering. What's the longest row of plants which he's able to water, if his speed is $u = 1\,\mathrm{m\cdot s^{-1}}$?

Hint *The speed with which water flows out of the hole is given by the formula $v = \mu s\sqrt{2gx}$ with a coefficient $\mu = 0.6$. where x is the current water level height.*

Marek was watering his garden.

We need to compute the time t needed for all the water to flow out of the bucket. It's given by

$$t = \int_0^h \frac{S}{v}\,\mathrm{d}x\,.$$

After substituting the formula for speed v, we get

$$t = \int_0^h \frac{S}{\mu s\sqrt{2gx}}\,\mathrm{d}x\,,$$

and after integration, we can find the time

$$t = \frac{2S\sqrt{h}}{\mu s\sqrt{2g}}\,.$$

Substituting the given values, we get $t \doteq 99.9\,\mathrm{s} \doteq 100\,\mathrm{s}$, which gives approximately $l = 100\,\mathrm{m}$ for the given speed of the gardener.

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Problem FoL.36 ... a merry encounter

In a vast empty space, there are six electrons very far from each other, located in the vertices of a regular hexagon. Each of them is moving towards the center of this hexagon with the speed $v = 1\,000\,\mathrm{m{\cdot}s^{-1}}$. How close to each other will two adjacent electrons get before the repulsive force separates them forever? *Mirek was inspired by Náry's struggles.*

If we consider the electrons to initially be very distant, we can neglect the forces between them, which means that their initial potential energy will be zero and the only present component of energy will be the kinetic energy of the electrons, which is

$$E_\mathrm{k} = 6 \cdot \frac{1}{2} m_\mathrm{e} v^2 \,,$$

where $m_\mathrm{e} = 9.1 \cdot 10^{-31}$ kg is the rest mass of an electron. The moment at which the electrostatic forces start to be significant is not really important for us, we only need to know that the forces affecting each electron will be of the same magnitude and pointing outwards from the center. Therefore, at some point, the electrons will stop in the vertices of a regular hexagon, which, obviously, will be smaller than the one in the beginning. There, their kinetic energy will be zero, and their potential energy will reach its maximum.

The potential energy of a system of point charges is equal to the sum of the potentials over each pair of charges. Generally,

$$E_\mathrm{p} = \frac{1}{2} \sum_{i=1}^{n} q_i \sum_{j=1, j \neq i}^{n} \frac{k q_j}{r_{ij}} \,,$$

where $k = (4\pi\varepsilon_0)^{-1}$ is Couloumb's constant, q is the point charge, r_{ij} is the distance between charges q_i and q_j, and n is the number of charges. If we label the distance between adjacent electrons r_1, the length of the shorter diagonal r_2 and the length of the longer diagonal r_3, and consider all q_i to be elementary charges, we can write

$$E_\mathrm{p} = 6 \cdot \frac{1}{2} k e^2 \left(\frac{2}{r_1} + \frac{2}{r_2} + \frac{1}{r_3} \right) .$$

The distances written as multiples of the side length r of the hexagon is

$$r_2 = \sqrt{3} r \,, \quad r_3 = 2r \,.$$

The potential energy has to be equal to the initial kinetic energy, which gives us the equation

$$\frac{3ke^2}{r} \left(2 + \frac{2}{\sqrt{3}} + \frac{1}{2} \right) = 3 m_\mathrm{e} v^2 \,, r = \frac{ke^2}{m_\mathrm{e} v^2} \left(2 + \frac{2}{\sqrt{3}} + \frac{1}{2} \right) .$$

Using the given values and the values of physical constants, we can evaluate the equations and get the result $r \doteq 9.3 \cdot 10^{-4}$ m.

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Problem FoL.37 ... Achilles and the tortoise

Consider an optical fibre which consists of a core, with diameter $d = 50\,\mu\mathrm{m}$ *and refractive index* $n_1 = 1.460$*, which is surrounded by cladding with refractive index* $n_2 < n_1$*. The rays propagate inside the core due to total internal reflection. A measurement was performed on a fibre with length* $l = 500\,\mathrm{m}$ *and it was estimated that the time dispersion (which is due to the rays propagating along different paths) at the receiving end was* $\Delta t = 20\,\mathrm{ns}$*. Determine the refractive index* n_2 *of the cladding. Assume that the speed of light in vacuum is* $c = 2.9979 \cdot 10^8\,\mathrm{m{\cdot}s^{-1}}$ *and that light is not dispersed inside the fibre.* *Michaleus.*

The ray of light that arrives first propagates along a straight path in the core, which corresponds to time $n_1 l/c$. The last ray is reflected from the boundary between the core and the cladding at the critical angle α. Considering the geometry of the problem, we can write $l' = l/\sin\alpha$. Let's consider the case of total reflection; then, $n_2 = n_1 \sin\alpha$, so if we know Δt, we can write

$$n_2 = n_1 \frac{l n_1}{c\Delta t + l n_1}\,.$$

Plugging in the numbers, we arrive at $n_2 \doteq 1.448$.

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Problem FoL.38 ... this problem sucks

Find the radius (in millimeters) of the largest lead ball which we are able to suck up using a vacuum cleaner with suction power $200\,\mathrm{W}$*. The machine is equipped with a plain hose with a circular cross-section of diameter* $6\,\mathrm{cm}$*. To simplify matters, assume that the ball is already hovering in the air high above the floor. Take the densities of air and lead to be* $\varrho = 1.2\,\mathrm{kg{\cdot}m^{-3}}$ *and* $\varrho_{\mathrm{Pb}} = 11\,340\,\mathrm{kg{\cdot}m^{-3}}$*, respectively.* *Kuba lost his pen-drive in a peculiar way.*

First, let us relate the suction power P of the vacuum cleaner to the speed v of the air entering the hose. If we denote by ϱ the density of the air and if we let $A = \pi d^2/4$ be the cross-sectional area of the hose (where d is the cross-sectional diameter), then the engine of the vacuum cleaner works to accelerate a mass $\mu = \varrho A v$ of the air per unit time, from rest to the speed of v. We then have

$$P = \frac{1}{2}\mu v^2 = \frac{1}{2}\varrho A v^3\,,$$

so

$$v = \left(\frac{2P}{\varrho A}\right)^{\frac{1}{3}}\,.$$

It remains to equate the magnitude of quadratic drag on the ball (we can check that the value of Re indeed suggests using quadratic drag) and gravity acting on the ball with density ϱ_{Pb}, radius r and drag coefficient C (since the ball hovers in the air far from any other objects, the situation is well approximated by turbulent flow past a rigid sphere with far-field velocity v). Therefore

$$\frac{4}{3}\pi\varrho_{\mathrm{Pb}} g r^3 \leq \frac{1}{2}\varrho C \pi r^2 \left(\frac{8P}{\varrho\pi d^2}\right)^{\frac{2}{3}}\,,$$

so we obtain

$$r \leq r_{\max} = \frac{3}{8g} C \left(\frac{8P}{\varrho \pi d^2} \right)^{\frac{2}{3}} \frac{\varrho}{\varrho_{\mathrm{Pb}}} \,.$$

Using $C = 0.5$, we have $r_{\max} \doteq 4.9\,\mathrm{mm}$.

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Problem FoL.39 ... bursting with rage

Kiki found a hair band and didn't know what to do with it, so she put it around Náry's head. The length of the unstretched band is $l_0 = 15\,\mathrm{cm}$, *its length when put around Náry's head is* $l = 55\,\mathrm{cm}$. *The cross-section of the band has a square shape with a constant side length of* $w = 2\,\mathrm{mm}$. *Let us naively assume that the band is behaving linearly according to Hooke's law, and has Young's modulus* $E = 50\,\mathrm{MPa}$. *What is the pressure (in* kPa*) crushing Náry's head? Take the head to be perfectly spherical and the band to be lying on its great circle.*

Watermelon gore by Mirek.

We can use the very simplifying assumption of linearity of the band – the force of tension in the band is

$$F = w^2 E \frac{l - l_0}{l_0} \,.$$

Now we need to find how much a small length element $\mathrm{d}l$ pushes on Náry's head. The angle corresponding to this element is ϑ. Both ends of this element are pulled tangentially with force $\boldsymbol{F}$, so these two forces add up vector-wise and the resulting force pushing on the head is

$$\mathrm{d}F_{\mathrm{r}} = 2F \sin \frac{\vartheta}{2} \,.$$

In polar coordinates, we can write the element length as $\mathrm{d}l = R\vartheta$, where R is the radius of the circle with circumference l. The contact area element is $\mathrm{d}S = wR\vartheta$, and using the circumference instead of the radius, we have

$$\mathrm{d}S = \frac{\vartheta l w}{2\pi} \,.$$

The pressure can be computed as the limit over a very small area, and it suffices to send the angle ϑ to 0

$$p = \lim_{\vartheta \to 0} \frac{\mathrm{d}F_{\mathrm{r}}}{\mathrm{d}S} = \lim_{\vartheta \to 0} \frac{2F \sin(\vartheta/2)}{\vartheta l w/(2\pi)} = \frac{2\pi F}{lw} = \frac{2\pi w^2 E (l - l_0)}{l l_0 w} = 2\pi w E \left(\frac{1}{l_0} - \frac{1}{l} \right) \,.$$

After plugging in the numbers, we get $p \doteq 3.05\,\mathrm{MPa}$. It is, however, reasonable to doubt the correctness of this result, since the material properties of the hair band are probably more complicated.

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Problem FoL.40 ... alternating-direct 2

Consider the circuit in the figure, the DC source has voltage $U_\mathrm{j} = 4.5\,\mathrm{V}$, the AC source has voltage amplitude $U_\mathrm{s} = 5\,\mathrm{V}$ and frequency $f = 50\,\mathrm{Hz}$. The resistor's resistance is $R = 100\,\mathrm{k\Omega}$ and the capacitor's capacity is $C = 10\,\mathrm{nF}$. What's the maximum charge on the capacitor, in nC?

Xellos still doesn't like broccoli.

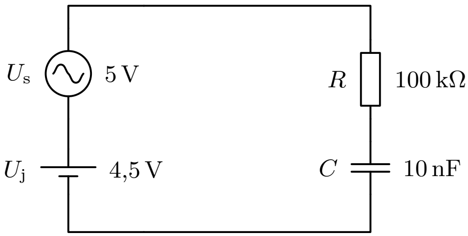

Fig. 6: Circuit diagram.

The two sources are again a trap – all elements of the circuit are linear, so we can view it as a superposition (sum) of two circuits – one with the DC and the other with the AC source. The charge on the capacitor is a superposition of charges in both cases. As we already know, in the circuit with the DC source, the capacitor carries a constant charge $Q_\mathrm{j} = CU_\mathrm{j}$.

The circuit with the AC source is harder to analyse, because we can't just ignore the resistor. In order to find the maximum charge on the capacitor, we need to find the maximum voltage on it. We'll express the immediate voltage u_c on the capacitor from the 2$^\mathrm{nd}$ Kirchhoff's law using the immediate voltages on the AC source u_s and on the resistor u_r as $u_\mathrm{c} = u_\mathrm{s} - u_\mathrm{r} = u_\mathrm{s} - jR$, where j is the immediate current through the circuit. The impedance of the circuit is $Z = R + 1/(\mathrm{i}\omega C)$ and we know that $j = u_\mathrm{s}/Z$, so

$$u_\mathrm{c} = u_\mathrm{s}\left(1 - \frac{R}{Z}\right) = u_\mathrm{s}\frac{1}{1 + \mathrm{i}R\omega C} = u_\mathrm{s}\frac{1}{1 + 2\mathrm{i}\pi f RC}$$

and the maximum value of U_c is

$$U_\mathrm{c} = U_\mathrm{s}\left|\frac{1}{1 + 2\mathrm{i}\pi f RC}\right| = \frac{U_\mathrm{s}}{\sqrt{1 + (2\pi f RC)^2}}\,.$$

Therefore, the maximum charge on the capacitor is

$$Q = C\left(U_\mathrm{j} + \frac{U_\mathrm{s}}{\sqrt{1 + (2\pi f RC)^2}}\right) \doteq 92.7\,\mathrm{nC}\,.$$

Note that in the circuit with the AC source, there's a phase shift between u_c and u_s, which causes the maximum of u_c to occur at a different time than the maximum of u_s.

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Problem FoL.41 ... ambulance

Verča wanted to pretend being an ambulance. She took a honking siren and started running from Karel towards the door at a speed $v = 7\,\mathrm{km{\cdot}h^{-1}}$. Karel, who watches the situation with interest, hears interference beats at a frequency $f = 6\,\mathrm{Hz}$. What is the frequency f_0 of the siren? The speed of sound in the air is $v_\mathrm{s} = 340\,\mathrm{m{\cdot}s^{-1}}$.

Verča is starting to go crazy from all that physics.

The beats that Karel is hearing are a superposition of two waves – one direct and one reflected from the door. The movement of the source causes Doppler's effect. Therefore, both frequencies differ from the frequency of the siren. The frequency f_1 is lower (the source is moving away from the observer) according to the relation for $v \ll v_\mathrm{s}$

$$f_1 = f_0 \left(1 - \frac{v}{v_\mathrm{s}}\right) .$$

The amplitude A which Karel is hearing can be described as

$$A(t) = A_0 \left(\cos\left(2\pi f_1 t\right) + \cos\left(2\pi f_2 t + \varphi\right)\right) ,$$

where A_0 is the amplitude of the original wave and φ is the phase shift of the first wave against the second. Using the trigonometric identity for the sum of cosines, we get

$$A(t) = 2A_0 \cos\left(\frac{2\pi(f_1 + f_2) + \varphi}{2} t\right) \cos\left(\frac{2\pi(f_1 - f_2) - \varphi}{2} t\right) .$$

The first cosine's frequency is $(f_1 + f_2)/2 = f_0$, which is too high; that means the beats come from the second cosine. The human ear can only perceive the absolute value of the amplitude. During one period of the cosine, two beats are generated (with opposite phase). Therefore, the frequency of beats is twice the frequency $|f_1 - f_2|/2$ of the second cosine. Hence

$$f = f_2 - f_1 = f_0 \frac{2v}{v_\mathrm{s}} ,$$

from which we isolate f_0

$$f_0 = f \frac{v_\mathrm{s}}{2v} .$$

For the given numeric quantities, we get $525\,\mathrm{Hz}$.

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Problem FoL.42 ... transmittors

Imagine that we have two plane-parallel mirrors and we know that each mirror transmits only a fraction of incident light. The first mirror transmits $t_1 = 2/3$ of incident light, the second only $t_2 = 1/3$. What fraction T of incident light passes through our system of two parallel mirrors? No energy is lost in the mirrors.

Karel pondered what "showing someone a mirror" meant in the former Communist Bloc.

First, we observe that we have been given transmission coefficients t_i and the reflection coefficients are simply $1 = t_i + r_i \Rightarrow r_i = 1 - t_i$, where i indicates the respective mirror.

Next, we mustn't forget that the light may bounce from one mirror to the other infinitely many times (theoretically), until its energy decreases to zero. To obtain the total coefficient of reflection, we must evaluate a geometric series

$$T = t_1t_2 + t_1r_2r_1t_2 + t_1r_2r_1r_2r_1t_2 + \cdots = t_1t_2\sum_{n=0}^{\infty}(r_1r_2)^i = t_1t_2\frac{1}{1-r_1r_2}\,.$$

The formula implies that the result does not depend on the order of the mirrors (which should not surprise us). Now, we can rewrite the reflection coeffcient using $r_i = 1 - t_i$ and obtain the coefficient

$$T = \frac{t_1t_2}{t_1+t_2-t_1t_2} = \frac{1}{\frac{1}{t_1}+\frac{1}{t_2}-1} = \frac{2}{7} \doteq 0.286\,.$$

The transmission coefficient of the system of two mirrors is 0.286.

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Problem FoL.43 . . . homemade maglev

Mišo adores trains and desperately wants to take a ride on a maglev - a train that uses magnetic levitation to move without touching the ground. In order to avoid travelling far abroad, he decided to build his own maglev at home. He plans to use a set of electromagnets in the shape of the letter U *to create the required magnetic field. The inductor is wound on a core and has* $N = 100$ *turns. The cross section of the core is* $S = 5\,\mathrm{cm}^2$ *and it's constant along its whole length. The length of the central line of force inside the inductor is* $l = 20\,\mathrm{cm}$*. The core has relative permeability* $\mu_\mathrm{r} = 1\,000$*. To test the strength of the magnets, Mišo places a steel girder next to both ends of one magnet, then connects the inductor to a current source and tries to pull the girder away. How large will be the current passing through the inductor if he needs a force of* $F = 100\,\mathrm{N}$ *to pull the girder away? Consider the permeability of air to be approximately the same as the permeability of vacuum* $\mu_0 = 4\pi\cdot 10^{-7}\,\mathrm{H\cdot m^{-1}}$*. Neglect the weight of the girder.*

Mirek tried to find a "practical" use for excercise problems.

We can express the force needed to pull a girder away as

$$F = \frac{\mathrm{d}E}{\mathrm{d}x}\,,$$

where E is the energy of the magnetic field of the inductor. The magnitude of magnetic induction can be derived from Ampère's law; we get

$$B = \mu_\mathrm{r}\mu_0\frac{NI}{l}\,,$$

where I is the current passing through the inductor. The energy density of magnetic field w outside the inductor is given by

$$w = \frac{B^2}{2\mu_0}\,.$$

Let's express a volume element as $\mathrm{d}V = 2s\mathrm{d}x$, then

$$\mathrm{d}E = w\mathrm{d}V = 2Sw\mathrm{d}x\,.$$

After dividing by dx

$$F = 2Sw = \frac{SB^2}{\mu_0}\,.$$

We are left with substituting for the magnetic field and expressing the current

$$I = \frac{l}{N\mu_\mathrm{r}}\sqrt{\frac{F}{\mu_0 S}} = 0.798\,\mathrm{A}\,.$$

The current in the inductor has to be $I = 0.798\,\mathrm{A}$.

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Problem FoL.44 ... accretion is too mainstream

According to some theories, there used to be a 10th planet of the Solar System called Phaeton between Mars and Jupiter. Its distance from the Sun is supposed to have been approximately $D_\mathrm{P} = 2.5\,\mathrm{au}$ and its radius $r_\mathrm{P} = 1\,000\,\mathrm{km}$. Determine the surface temperature of Phaeton after a radiation equilibrium is established, if its Bond albedo is similar to that of Earth, $A = 0.3$ (and so is emissivity, which is approximately 1). Phaeton is supposed to have existed at a time when the Sun had surface temperature $T_\mathrm{S} = 5\,000\,\mathrm{K}$ and radius $r_\mathrm{S} = 6\cdot 10^5\,\mathrm{km}$.

Mirek has been thinking about moving somewhere else.

The main source of radiated energy in the Solar System is the Sun. We'll compute its radiative power using the Stefan-Boltzmann law

$$P_\mathrm{S} = 4\pi r_\mathrm{S}^2 \sigma T_\mathrm{S}^4\,,$$

where $\sigma = 5.7\cdot 10^{-8}\,\mathrm{W\cdot m^{-2}\cdot K^{-4}}$ is the Stefan-Boltzmann constant. The power incident on Phaeton is proportional to the surface covered by the planet on a sphere of radius D_P, which is

$$P_\mathrm{inc} = P_\mathrm{S}\frac{\pi r_\mathrm{P}^2}{4\pi D_\mathrm{P}^2} = \pi r_\mathrm{S}^2 \sigma T_\mathrm{S}^4 \frac{r_\mathrm{P}^2}{D_\mathrm{P}^2}\,.$$

The Bond albedo determines the fraction of power that's reflected from the surface of the planet, so only power

$$P = P_\mathrm{inc}(1-A)$$

is actually absorbed. In the state of radiation equilibrium, the absorbed power must be equal to the emitted power

$$P_\mathrm{P} = 4\pi\sigma r_\mathrm{P}^2 T_\mathrm{P}^4\,,$$

radiated by the planet. We neglected the emissivity of Earth here, since it's sufficiently close to 1. We get the equation

$$\pi r_\mathrm{S}^2 \sigma T_\mathrm{S}^4 \frac{r_\mathrm{P}^2}{D_\mathrm{P}^2}(1-A) = 4\pi\sigma r_\mathrm{P}^2 T_\mathrm{P}^4\,,$$

from which we'll express the desired temperature

$$T_\mathrm{P} = T_\mathrm{S}\sqrt[4]{\frac{r_\mathrm{S}^2}{4D_\mathrm{P}^2}(1-A)}\,.$$

Numerically, we get $T_\mathrm{P} \doteq 130\,\mathrm{K}$.

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Problem FoL.45 ... wavy electron

What wavelength will an electron have, if we speed it up from rest with electric field over a potential difference of $60\,\mathrm{kV}$*?* *Kiki and her introduction to quantum physics for dogs.*

Let us use de Broglie's relation $\lambda = h/p$, where h is the Planck constant and p the momentum of the electron, which we have to determine. The kinetic energy which the electron obtains when crossing the given potential difference $\Delta\varphi$ is $E_\mathrm{k} = e\Delta\varphi$, where e is the elementary charge. The total energy of the electron with rest mass m_e will then be $E = E_0 + E_\mathrm{k}$, where $E_0 = m_\mathrm{e}c^2$ is its rest energy. The relation between energy E and momentum p is $(cp)^2 + E_0^2 = E^2$, from which we can express $p = \sqrt{(m_\mathrm{e}c + e\varphi/c)^2 - (m_\mathrm{e}c)^2}$ and then the wavelength as

$$\lambda = \frac{h}{\sqrt{\left(m_\mathrm{e}c + \frac{e\varphi}{c}\right)^2 - (m_\mathrm{e}c)^2}} = 4.87 \cdot 10^{-12}\,\mathrm{m}\,.$$

The electron is almost located at a point and its wavelength decreases with increasing voltage; maybe in contradiction with intuition, a faster particle isn't "wider".

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Problem FoL.46 ... rising bubble

Having had a great time doing our laundry in a sink, we managed to release a number of soap bubbles into the air. Some of them have burst after flying around for a short while, but a number of them happened to ascend relatively high up – the most successful guy reached the altitude of $H = 1\,500\,\mathrm{m}$ *above ground level. How many times did the volume of this bubble increase during its ascent? Use the ideal gas law and the equation of hydrostatic equilibrium applied to the atmosphere in order to find the pressure as a function of altitude above ground level. Assume that there is a linear relation of the form* $T(h) = T(0) - kh$ *between the temperature and the altitude, where* $k = 0.006\,5\,\mathrm{K{\cdot}m^{-1}}$*. The atmospheric pressure at ground level is* $p(0) = 101\,\mathrm{kPa}$*, the temperature at ground level is* $T(0) = 25\,^\circ\mathrm{C}$*. The molar mass of air is* $M = 29.0\,\mathrm{g{\cdot}mol^{-1}}$*. Assume that the pressure inside and outside the bubble is the same.*

Mirek doing calculations and laundry at the same time.

Substituting from the ideal gas law

$$\varrho = \frac{pM}{RT}$$

into the differential form of the equation of hydrostatic equilibrium

$$\frac{\mathrm{d}p}{\mathrm{d}h} = -\varrho g\,,$$

we get a separable first-order ODE

$$\frac{\mathrm{d}p}{p} = -\frac{gM}{RT}\mathrm{d}h = -\frac{gM}{R(T(0) - kh)}\mathrm{d}h\,.$$

Integrating (and taking into account the appropriate boundary conditions), we obtain an expression for the ambient pressure p as a function of altitude h above ground

$$p = p(0)\left(1 - \frac{kh}{T(0)}\right)^{gM/Rk}\,.$$

Finally, using the ideal gas law $pV/T = \text{const}$, we find

$$\frac{V(h)}{V(0)} = \frac{T(h)p(0)}{T(0)p(h)} = 1.15\,.$$

The volume of the bubble increased 1.15 times.

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Problem FoL.47 ... a charged spiral

Consider a charged non-conducting spiral parametrised in 2D cartesian coordinates as

$$\{[l_0 t \sin \ln t, l_0 t \cos \ln t], t > 0\}\,.$$

Assume that the linear charge density ϱ along the spiral can be expressed as a function of t, namely $\varrho(t) = \varrho_0 t \exp(-t^2)$. Find the value of electric potential at the origin and express it as a multiple of ϱ_0/ε_0. The spiral is placed in vacuum. *Due to Janči.*

Let us first find an expression for the length of an infinitesimal element of the spiral corresponding to a change $\mathrm{d}t$ in the parameter t. Using the Pythagorean theorem, we obtain

$$\begin{aligned} \mathrm{d}l &= \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = l_0 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\ &= l_0 \sqrt{(\sin \ln t + \cos \ln t)^2 + (\cos \ln t - \sin \ln t)^2}\mathrm{d}t = \sqrt{2} l_0\,\mathrm{d}t\,. \end{aligned}$$

The distance between the origin and a point on the spiral corresponding to a given value of t is simply $r(t) = l_0 t$, so the potential can be found by direct integration as

$$\varphi = \int_0^\infty \frac{1}{4\pi\varepsilon_0} \frac{\varrho(t)\mathrm{d}l}{r(t)} = \frac{\sqrt{2}\varrho_0}{4\pi\varepsilon_0} \int_0^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t = \frac{\sqrt{2}\varrho_0}{4\pi\varepsilon_0} \frac{\sqrt{\pi}}{2} = \frac{\sqrt{2}}{8\sqrt{\pi}} \approx 0.099\,7\,\frac{\varrho_0}{\varepsilon_0}\,.$$

The potential in the desired units is $0.099\,7\,\varrho_0/\varepsilon_0$.

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Problem FoL.48 ... washing the dishes

Verča likes to wash the dishes. When she's done, she takes a little boat and places it on the surface of water mixed with detergent. The boat is a small wooden board with thickness $b = 0.5\,\mathrm{cm}$, *length* $l = 10\,\mathrm{cm}$ *and width* $w = 5\,\mathrm{cm}$. *Its length is parallel to the axis x. The surface tension in the x-direction is given by $\sigma(x) = \sigma_0 + xs$ and in the y-direction by $\sigma(y) = \sigma_0$, where* $\sigma_0 = \text{const}$ *and s is the gradient of surface tension. Determine the intial acceleration of boat after being placed on the water surface. The density of wood is* $\varrho = 800\,\mathrm{kg{\cdot}m^{-3}}$, $\sigma_0 = 30\,\mathrm{mN{\cdot}m^{-1}}$, $s = 80\,\mathrm{mN{\cdot}m^{-2}}$. *If the acceleration is in the direction of increasing surface tension, it's positive; if it's in the opposite direction, it's negative. The contact angle of wood and water is $\beta = 45^\circ$. The inclination of the boat is negligible.*

Mirek watched students washing the dishes during the summer camp.

The problem is heavily simplifed by the fact that the boat is rectangular and its length is parallel to the gradient of the surface tension. Thus, the resulting force acting on the boat is simply the difference of forces acting on the front and the rear side of the ship (the lateral forces compensate for each other). The forces must be multiplied by the factor $\sin\beta$, because their vertical components compensate for the gravity of the boat.

The surface tension represents the propelling force per element of length; in our case, it is the length of the shorter side w. The magnitude of the propelling force is

$$(F(l) - F(0))\sin\beta = ((\sigma_0 + (x+l)s) - (\sigma_0 + xs))\, w \sin\beta = wls\sin\beta\,,$$

the result is positive, so the boat moves in the direction of increasing surface tension. The mass of the boat is $m = \varrho wlb$, so

$$a = \frac{s\sin\beta}{\varrho b}\,.$$

Numerically, $a \doteq 0.014\,\mathrm{m{\cdot}s^{-2}}$.

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Problem FoL.49 ... amplifying

Find the gain of the circuit in the figure. The gain is defined as $g = U_{\mathrm{output}}/U_{\mathrm{input}}$ (including the sign, the voltages are calculated with respect to the ground). All resistors in the circuit have resistivity $R = 10\,\mathrm{k\Omega}$. Assume that the op-amp is ideal.

Hint *To solve this problem, you will only need Ohm's law, Kirchhoff's first law and the following information: The triangle symbol in the circuit is an op-amp (operational amplifier). It has two inputs (− and +) and one output (the third vertex of the triangle), and follows these rules:*

- *there is no current going in or out of any of the inputs,*
- *the op-amp sets the voltage at its output so that the voltage difference between its inputs is zero.*

Pikoš likes playing with complicated circuits.

The input − is called inverting, the input + non-inverting. This use of the op-amp (connecting the output back to the input) is called feedback.

First, we need to find the voltages on the two inputs of the op-amp. Since no current flows into them, the current through R_5 is zero and, using Ohm's law, so is the voltage across it. We write $U_5 = 0\,\mathrm{V}$. This makes the voltage with respect to the ground (junction E) on the inverting input zero. Because the op-amp wants to keep the voltage difference between its inputs equal to zero, the voltage on the non-inverting input (again, with respect to the ground) is also zero.

Now we know that the voltage between the junctions A and E is zero, the voltage across the resistor R_1 is $U_1 = U_{\mathrm{input}}$ and the current through it is $I_1 = U_1/R_1 = U_{\mathrm{input}}/R_1$. Because from the junction A, no current can flow into the op-amp, the current through the resistor R_2 is $I_2 = I_1$. We can then calculate the voltage across the resistor R_2 as $U_2 = R_2 I_2 = U_{\mathrm{input}} R_2/R_1$. Because the voltage between the junction A and the ground is zero, the voltage between the junction B and the ground is $U_4 = U_{\mathrm{input}} R_2/R_1$, and the potential is lower in the junction B.

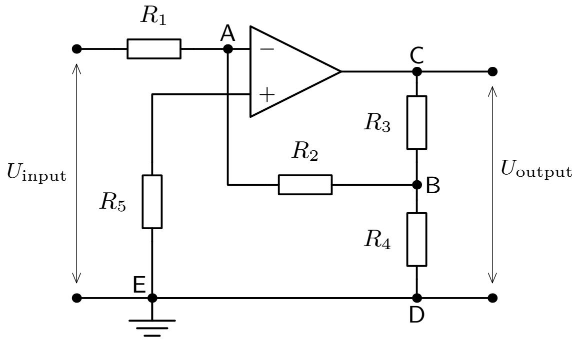

Fig. 7: Circuit scheme.

Because the potential in the junction B is lower that the potential in the junction D, the current flows from D to B and its magnitude is $I_4 = U_4/R_4 = U_{\mathrm{input}}R_2/(R_1R_4)$.

The currents flowing into the junction B are I_2 through the resistor R_2 and I_4 through the resistor R_4. Therefore, using the first Kirchhoff's law, the outgoing current, through the resistor R_3, is $I_3 = I_2 + I_4 = U_{\mathrm{input}}[1/R_1 + R_2/(R_1R_4)]$, the voltage across it is $U_3 = R_3I_3 = U_{\mathrm{input}}R_3[1/R_1 + R_2/(R_1R_4)]$ and, because of the direction of the current, the potential is lower in C than in B.

The voltage on the output is now easily computed as $U_{\mathrm{output}} = -(U_4 + U_3)$ (the minus sign is there because the potential is lower in C than in B, which has lower potential than D), so

$$U_{\mathrm{output}} = -U_{\mathrm{input}}\left[\frac{R_2}{R_1} + R_3\left(\frac{1}{R_1} + \frac{R_2}{R_1R_4}\right)\right] ,$$

and the gain is

$$g = \frac{U_{\mathrm{output}}}{U_{\mathrm{input}}} = -\frac{R_2R_4 + R_3R_4 + R_2R_3}{R_1R_4} ;$$

if all the resistors have the same resistances, the gain is $g = -3$.

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Problem FoL.50 ... watering

In a pool full of water, there is a vertical tube with internal cross-section $S_1 = 4\,\mathrm{cm}^2$*. On its top, it opens into a cylindrical container with a few small holes on the perimeter. The whole construction rotates and the cylindrical container is radially partitioned, so that the water rotates with the container. The cylinder has radius* $r = 10\,\mathrm{cm}$ *and the total cross-section of the holes is* $S_2 = 5\,\mathrm{mm}^2$*. At first, the holes are closed and the whole tube and cylinder are filled with water and rotated with an angular speed* $\omega = 25\,\mathrm{rad{\cdot}s^{-1}}$*. Then, the holes are opened, so that water can flow out. Determine the speed* v *of the water flowing out of the holes with respect to the laboratory reference frame (which doesn't rotate with the container). The air pressure is* $p_0 = 1\,015\,\mathrm{hPa}$*, the acceleration due to gravity is* $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$ *and the holes in the cyliner are* $h = 20\,\mathrm{cm}$ *above the surface of the pool. Consider the density of water to* be $\varrho = 1\,000\,\mathrm{kg{\cdot}m^{-3}}$.

Faleš was inspired by the Estonian-Finnish olympiad, which had a partially wrong solution.

First, let's determine the pressure (with respect to the ambient atmospheric pressure) that will be pushing the water out of the container. Gravity will pull the water with pressure $p_\mathrm{g} = -\varrho g h$. The rotation of the cylinder will, however, push the water out, increasing the pressure near the perimeter and decreasing it near the center, sucking more water in. The centrifugal force acting on an element with a mass m at a distance r from the center is

$$F_\mathrm{o} = m\omega^2 r\,.$$

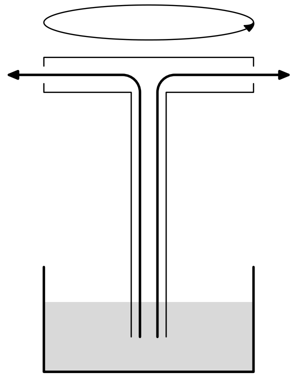

Fig. 8: Scheme of the water pump.

The pressure difference is effectively the potential energy change of a unit volume of water. Thus, we need to find the potential to this force, which we do by integrating (we may neglect the radius of the tube, so we start at the radius 0)

$$U_\mathrm{c} = \int_0^r m\omega^2 r \mathrm{d}r = \frac{1}{2} m\omega^2 r^2\,.$$

The pressure is then

$$p_\mathrm{c} = \frac{1}{2}\varrho\omega^2 r^2\,.$$

The total pressure difference from the top of the tube to the perimeter of the cylinder is then

$$p = p_\mathrm{g} + p_\mathrm{c} = -\varrho g h + \frac{1}{2}\varrho\omega^2 r^2\,.$$

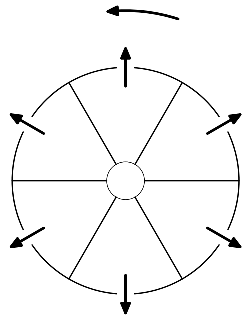

Fig. 9: Top view on the water reservoir.

The speed of the water that's flowing out is given by Bernoulli's equation

$$\frac{1}{2}\varrho v_\mathrm{r}^2 = p = \frac{1}{2}\varrho\omega^2 r^2 - \varrho g h\,.$$

So

$$v_\mathrm{r}^2 = \omega^2 r^2 - 2gh$$

is the square of the speed in the frame of reference rotating with the container. To get to the laboratory frame, we need to transform back. Luckily, the velocity is given by

$$\boldsymbol{v}_\mathrm{i} = \boldsymbol{v}_\mathrm{r} + \boldsymbol{\omega} \times \boldsymbol{r}$$

and the two terms on the RHS are perpendicular, so we can just use the Pythagorean theorem to add the velocities

$$v_\mathrm{i}^2 = v_\mathrm{r}^2 + (\omega r)^2\,,$$

so

$$v_\mathrm{i} = \sqrt{2\left(\omega^2 r^2 - g h\right)}\,.$$

Numerically, this is $v_\mathrm{i} \doteq 2.9\,\mathrm{m{\cdot}s^{-1}}$.

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Problem M.1 ... submarine

Michal is planning a journey from water level at the Mariana Trench (0 m above sea level) down to the seabed (10 971 m below sea level). What difference in hydrostatic pressure (in MPa) will Michal's submarine measure? Assume that water density doesn't change with depth. Consider the constants $\varrho_{\mathrm{H_2O}} = 1\,000\,\mathrm{kg\cdot m^{-3}}$ and $g = 9.81\,\mathrm{m\cdot s^{-2}}$.

Zuzka was thinking about a submarine.

Hydrostatic pressure p in depth h satisfies

$$p = \varrho g h\,,$$

where ϱ is water density. The pressure at the water level is p_2 and the pressure at the seabed is p_1. Then, the pressure difference satisfies

$$\Delta p = p_1 - p_2 = \varrho g h_1 - \varrho g h_2 = \varrho g (h_1 - h_2)\,.$$

After substitution, we get that the magnitude of change in hydrostatic pressure is 110 MPa.

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Problem M.2 ... wet FYKOS-bird

The FYKOS-bird is riding on a tricycle with speed $v = 10\,\mathrm{km\cdot h^{-1}}$, when, to his displeasure, it starts raining. Not only is he getting wet, but he also has to increase his power output in order to keep moving at the same speed. Assume that the surface exposed to the uniform rain is $S = 0.5\,\mathrm{m^2}$ and that on average, 2 droplets fall on each $\mathrm{cm^2}$ over time $\Delta t = 1\,\mathrm{s}$. One droplet weighs $m = 0.1\,\mathrm{g}$. The rain falls straight down, the bird is moving in a straight horizontal line at constant speed. Neglect the fact that the FYKOS-bird has to move with respect to the tricycle. By how much does he need to increase his output power? *Faleš got wet.*

The bird needs to compensate for the momentum he transfers to the droplets in the horizontal direction. Let's denote by S_0 the area element $1\,\mathrm{cm^2}$, on which $n = 2\,\mathrm{s^{-1}}$ droplets per time fall.

The momentum which the bird has to transfer to the droplets over time Δt then is

$$\Delta p = v \Delta M = v n m \frac{S}{S_0} \Delta t\,.$$

Power is work over time, where we can express work as force over distance and power as force times distance over time, or as force times speed

$$\Delta P = \frac{\Delta W}{\Delta t} = \frac{sF}{\Delta t} = v \frac{\Delta p}{\Delta t} = v^2 n m \frac{S}{S_0}\,.$$

Numerically, we get $\Delta P \doteq 7.7\,\mathrm{W}$.

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Problem M.3 ... water pump

A tube of diameter $r = 2\,\mathrm{cm}$, bent to the shape of the letter L*, has one end submerged in a water tank. Its lower part is parallel to the water surface and the other part sticks out from the water vertically. The tube moves in the direction of the line connecting the bend and the submerged end, with speed $v = 2\,\mathrm{m{\cdot}s^{-1}}$. How high above the water surface (in* cm*) in the tank will the water in the tube rise? Consider the acceleration due to gravity to be $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.*

Faleš saw a nice picture of a bent tube.

By Bernoulli's equation

$$\frac{1}{2}\varrho v^2 + p = \mathrm{const}\,.$$

The pressure against the water flowing into the tube caused by the water rise in the tube is

$$p = -h\varrho g\,.$$

Overall, we get the elevation h as

$$h = \frac{v^2}{2g}\,.$$

Numerically, $h \doteq 20\,\mathrm{cm}$.

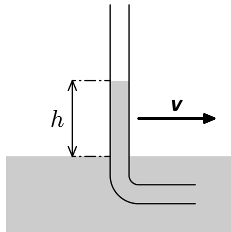

Fig. 10: Scheme of the pipe.

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Problem M.4 ... rocket bird

The Fykos-bird's wings hurt, so he decided to use physics and build a rocket-propelled backpack. During ignition, the backpack consumes $100\,\mathrm{g}$ of fuel per $1\,\mathrm{s}$ and exerts a $50\,\mathrm{N}$ propulsion force. What's the speed of exhaust gases with respect to the backpack? The bird with the jet-pack is much heavier than the consumed fuel.

Faleš's legs hurt.

The law of momentum conservation gives for the momentum changes of the bird and gas over time $\varDelta t$

$$m\frac{\varDelta v}{\varDelta t} = -\frac{\varDelta m}{\varDelta t}v_\mathrm{r}\,,$$

where v_r is the speed of gases with respect to the backpack, m is the mass of the propelled body (Fykos-bird + the backpack), $\varDelta v$ is the change of its speed, $\varDelta m$ the mass of gases emitted over time $\varDelta t$. The change of momentum over time is also the propulsion force

$$F = m\frac{\varDelta v}{\varDelta t}\,,$$

so we find the speed of gases as

$$v_\mathrm{r} = \frac{F\varDelta t}{\varDelta m}\,.$$

Numerically, $v_\mathrm{r} = 500\,\mathrm{m{\cdot}s^{-1}}$.

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Problem E.1 ... trinity

The resistance R_x is equal to the resistance of the whole circuit. The resistances R_1 and R_2 are the same. Compute the value of R_x as a multiple of $R = R_1 = R_2$.

Faleš liked the golden ratio.

We just have to write down the formula for the combination of resistors, or

$$xR = \frac{R\,(xR + R)}{R + (xR + R)}\,,$$

where x is the sought multiple of R, i. e., $xR = R_x$.

Solving this quadratic equation, we get two roots, and the positive one is $x \doteq 0.62$ (the golden ratio).

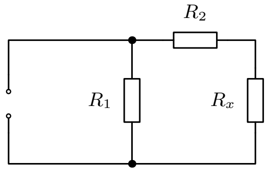

Fig. 11: Circuit diagram.

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Problem E.2 ... Náry brawls with numbers

An electromagnetic plane wave is travelling through vacuum. We have measured the values of electric intensity $\boldsymbol{E}$ and magnetic induction $\boldsymbol{B}$ at a particular place in space. The resulting values are $\boldsymbol{E} = (2, 3, 1)\,\mathrm{V{\cdot}m^{-1}}$, $\boldsymbol{B} = (5, -3, -1)\,\mathrm{T}$. We are interested in the direction in which the wave is travelling. Compute this direction, normalize the direction vector (multiply it by a positive number such that the resulting vector has unit length) and send us sum of its three components.

Náry was rotating while being normalized by field theory knowledge.

Since it's a plane wave, the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ are orthogonal. Moreover, both of them are orthogonal to the direction $\boldsymbol{n}$ in which the wave travels. In three dimensions, it holds true that $\boldsymbol{E} \times \boldsymbol{B} = \alpha \boldsymbol{n}$, where α is a positive number in the units of $\mathrm{m^{-1}{\cdot}s^5{\cdot}kg^{-2}{\cdot}A^2}$ – the number inverse to the normalization constant. The cross product gives us a vector $(0, 7, -21)\,\mathrm{m^{-1}{\cdot}s^5{\cdot}kg^{-2}{\cdot}A^2}$. We are left with finding the normalization constant; it is the inverse to the length of our vector – to the value $\sqrt{7^2 + (-21)^2}$. We divide our vector by this number and get the vector $\left(0, 1/\sqrt{10}, -3/\sqrt{10}\right)$. The sum of its components after rounding is -0.63.

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Problem E.3 ... Náry works hard

When he was supposed to move a telescope, Náry rather moved a charge $Q = 2\,\mathrm{C}$ from infinity to a place with electric potential $\varphi = 2\,\mathrm{kV}$. How much work (in mJ) did he do? Consider the potential at infinity to be zero.

Náry and Faleš at the theoretical physics camp.

Because the potential at infinity is zero and work is given as the difference of potentials times the charge, the work done by Náry is $W = Q\varphi \doteq 4.0 \cdot 10^6\,\mathrm{mJ}$.

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Problem E.4 ... capacitOr

A dielectric medium between two circular plates of a capacitor has two layers. The first one is air, with thickness $d_1 = 2\,\mathrm{mm}$, and the second is acrylic glass with thickness $d_2 = 4\,\mathrm{\mu m}$. Determine the capacity of the capacitor (in pF), if the area of each plate is $S = 2\,\mathrm{dm}^2$. The permittivity of vacuum is $\varepsilon_0 = 8.854 \cdot 10^{-12}\,\mathrm{C^2{\cdot}N^{-1}{\cdot}m^{-2}}$.

Faleš couldn't read a number in his old notebook, so he changed the dimensions.

This arrangement is equivalent to two capacitors in series, where capacities combine through inverse values

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}\,.$$

From this, we can express

$$C = \frac{C_1 C_2}{C_1 + C_2}\,.$$

The capacity of one capacitor is

$$\frac{\varepsilon_0 \varepsilon_\mathrm{r} S}{d}\,,$$

where ε_r is the relative permittivity of the medium in it. The permittivity of air is almost exactly equal to the permittivity of vacuum, so its relative permittivity is 1. The resulting capacity is then

$$C = \frac{\varepsilon_0 \varepsilon_\mathrm{r,p} S}{\varepsilon_\mathrm{r,p} d_1 + d_2}\,.$$

It looks like we are missing the relative permittivity of acrylic glass $\varepsilon_\mathrm{r,p}$. However, if we realize that d_2 is smaller than d_1 by three orders of magnitude, we find out that the capacity of the acrylic glass capacitor has only a small effect on the result (especially when rounded to one significant digit). The resulting capacity is therefore given by the capacity of the air capacitor, which is something we could have concluded at the beginning

$$C \approx C_1 = \frac{\varepsilon_0 S}{d_1}\,.$$

Numerically, we have $C \doteq 90\,\mathrm{pF}$.

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Problem X.1 ... we're expanding

What amount of work (in kJ) is done by an ideal gas during isothermal expansion, if its initial volume is $V_1 = 10\,\mathrm{dm}^3$ and initial pressure is $p_1 = 1.0\,\mathrm{MPa}$? The final pressure of the gas is $p_2 = 100\,\mathrm{kPa}$.

Marek doesn't remember where he got it from.

The gas expands to a certain volume V_2. The work done by the gas can be expressed as

$$W = \int_{V_1}^{V_2} p\,\mathrm{d}V\,. \tag{4}$$

An isothermal process is described by Boyle's law, so

$$p_1 V_1 = pV \quad \Rightarrow \quad p = \frac{p_1 V_1}{V}\,. \tag{5}$$

Substituting from (5) to (4) yields

$$W = \int_{V_1}^{V_2} \frac{p_1 V_1}{V}\, \mathrm{d}V = p_1 V_1 \ln \frac{V_2}{V_1}\,. \tag{6}$$

Now, we will replace V_2/V_1 with the pressure ratio obtained using Boyle's law

$$\frac{V_2}{V_1} = \frac{p_1}{p_2}\,. \tag{7}$$

Finally, with the aid of equation (7), we express the work and compute its numerical value

$$W = p_1 V_1 \ln \frac{p_1}{p_2} \doteq 23\,\mathrm{kJ}\,.$$

The amount of work done by the gas is $W = 23\,\mathrm{kJ}$.

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Problem X.2 ... fiat lux

Every time Domča gets all sad and weary in the evening, all she needs to do is switch her lamp on and the incoming photons always cheer her up. What is the number of photons emitted by a 100-watt lamp over a period of one second? Assume that the lamp emits yellow light with a wavelength of $580\,\mathrm{nm}$ with $10\,\%$ efficiency and the photons with other wavelengths are considered losses. *Kiki hates transitions to daylight saving.*

The total energy E_1 available for conversion into yellow photons is $E_1 = Pt\eta$, where $P = 100\,\mathrm{W}$ is the input power given in the problem, $t = 1\,\mathrm{s}$ is the time interval over which we count the emitted photons and $\eta = 0.1$ is the efficiency of converting the input power into radiative power of yellow photons. The energy of one photon is well-known to be $E = hf = hc/\lambda$, where f is its frequency, h is the Planck's constant, c is the speed of light and $\lambda = 580\,\mathrm{nm}$ is the wavelength which we assume that our photons have. The number of photons radiated by the lamp over an interval of one second is therefore determined as

$$N = \frac{Pt\eta\lambda}{hc}\,,$$

which gives approximately $2.9 \cdot 10^{19}$ photons.

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Problem X.3 ... Christmas

For a three-dimensional object, a line `o` *is called an n-fold axis of symmetry if the object does not change upon rotation around this axis by an angle $2\pi/n$. For example, a regular heptagon (regular planar polygon with 7 edges) has one 7-fold axis of symmetry, perpendicular to the heptagon and passing through its center.*

Janči was decorating cookies for Christmas. One of them was shaped like a sphere. Janči divided the surface into eight equal parts and coloured them alternately, such that the coloured *parts of the surface touched only in vertices. How many 3-fold axes of symmetry does this cookie have?* *Janči was helping in the kitchen.*

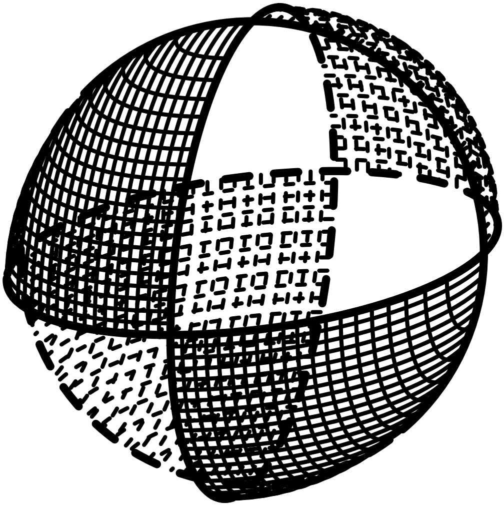

Fig. 12: Christmas cookie

Each axis has to pass through the center of the sphere, because otherwise, the rotation would move the sphere around.

Look at the picture – the desired axes have to pass through centres of opposite *curved triangles*, which means there are 4 distinct axes. More precisely, the sphere belongs to the point group T_{d}.

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Problem X.4 ... antichemical

One of the rules governing the filling of electron orbitals in atoms is Pauli's exclusion principle, which says that two electrons can never be in the same state. This gives us the usual rule of two electrons in one orbital (with the same quantum numbers n, l and m), because they can differ by the orientations of their spins (this is the fourth quantum number m_{s}).

Imagine that we changed this rule and now, two electrons can be in the same state, but not three. How many valence *electrons would a neutral sulphur atom have, if the orbitals were still being filled in the same order as for normal electrons.* *Lada was thinking about fermions.*

The energetic order of orbitals is $1s$, $2s$, $2p$ (the other ones are empty in our sulphur with modified electrons). The first orbital contains 4 electrons, with spins

$$1s: \uparrow\uparrow\downarrow\downarrow\,.$$

The valence electrons are those in orbitals with $n = 2$, i.e. all the remaining ones. Neutral sulphur has 16 electrons in total, which gives us 12 valence electrons.

More precisely, the orbital $2s$ would look the same as $1s$ and (three) orbitals $2p$ would contain the remaining 8 electrons, although $3 \times 4 = 12$ electrons in total could fit into them.

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