Solutions of Physics Brawl Online 2023

Problem 1 ... rotating a force 3 points

There are two forces $F_1 = 2.0\,\mathrm{N}$ and $F_2 = 1.0\,\mathrm{N}$ acting on a point of mass. What is the angle between them if their resultant is the same magnitude as the larger of the forces, $F = F_1$?

May the Force be with you!

It is essential to map out the situation well.

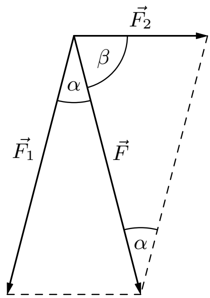

Figure 1: Force decomposition.

From the picture, we can see that we get two triangles with known side lengths. The angle we are looking for is $\alpha + \beta$. Since the triangles are isosceles, $\alpha + 2\beta = 180^\circ$, so we just need to calculate α because $\alpha + \beta = \alpha/2 + 90^\circ$.

We calculate the angle α using the law of cosines as

$$\cos\alpha = \frac{F_1^2 + F^2 - F_2^2}{2F_1F} = \frac{7}{8}\,.$$

From that, we get

$$\alpha/2 + 90^\circ = \arccos\left(\frac{7}{8}\right)/2 + 90^\circ = 104.5^\circ\,.$$

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Problem 2 ... five-second rule 3 points

You may have heard that if you drop your food on the floor but pick it up within five seconds, it will not be heavily contaminated with bacteria. Let us consider the following case. You drop a circular snack with a diameter of $4\,\mathrm{cm}$ on the ground. The bacteria from the floor will immediately stick to it. However, according to the common rule, this should not matter. Therefore, let us assume that most of the bacteria only arrive at the food from the vicinity of the snack during those five seconds. What would their velocity have to be for their numbers to multiply tenfold on a snack in five seconds? The surface density of bacteria on the ground is homogeneous. *Jarda always blows off the fallen food so that he can eat with a peace in mind.*

Let us assume that bacteria are a smart spieces, and as soon as the food falls on the ground, they instantly start moving towards it. Within five seconds, the bacteria can reach the snack from a distance $vt + r$, where v is their speed, $t = 5\,\mathrm{s}$ and $r = 2\,\mathrm{cm}$ is the radius of the snack.

In the beginning, $n_1 = \sigma \pi d^2/4$ of bacteria is stuck on the snack, where d is the diameter of the snack and σ is the surface density of bacteria on the floor, which is considered constant in the surroundings of the fallen food. To increase the number of bacteria tenfold, the area of the circle must increase by a factor of ten, which corresponds to the radius

$$R = \sqrt{10}r = vt + r\,,$$

from where the required bacteria speed is

$$v = \frac{R-r}{t} = \left(\sqrt{10} - 1\right)\frac{d}{2t} = 8.6\,\mathrm{mm{\cdot}s^{-1}}\,,$$

which is unrealistically high compared to their usual speed, which is at most in the order of tens of micrometers per second. Moreover, we must point out that the five-second rule has never been experimentally proven.

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Problem 3 ... game of tag — 3 points

Two cars are driving on a road parallel to each other in the same direction. The trajectories of the cars are $\xi = 1.5\,\mathrm{m}$ apart. Nicolas is $d = 3\,\mathrm{m}$ away from the trajectory of the first car, which is moving at $v_1 = 55\,\mathrm{km{\cdot}h^{-1}}$. What is the velocity of the second car if it always stays hidden behind the first car from Nicolas's point of view? We approximate the cars as point masses. *Nicolas waited for far too long at the bus stop.*

We will solve this problem using the geometry in the figure.

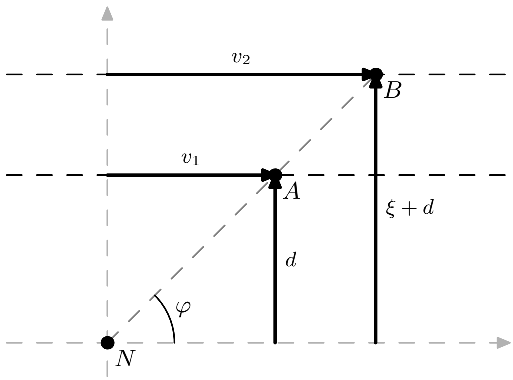

Figure 2: Sketch of the situation.

In the figure, instead of using the length of the sides, we have used vectors to represent the change in position of the point masses compared to the origin. We have set the origin at the point where Nicolas is standing, and we will use symmetry only for the case where both cars were directly in front of Nicolas at time t_0 and they have moved over the distance $s = v \cdot (t_1 - t_0)$, at t_1 to points A and B.

Consequently, we can notice that the triangles are similar based on the AA (Angle-Angle) theorem about the similarity of triangles. The similarity of the triangles is due to the angle φ and the right angle to the x-axis. The similarity of triangles means that the ratios of the hypotenuses are the same,[1] and thus we have

$$\frac{v_1 \cdot (t_1 - t_0)}{d} = \frac{v_2 \cdot (t_1 - t_0)}{d + \xi}$$
$$v_2 = \frac{d + \xi}{d} v_1 \,.$$

After substituting the values $v_1 = 55\,\mathrm{km{\cdot}h^{-1}}$, $\xi = 1.5\,\mathrm{m}$, $d = 3\,\mathrm{m}$, we got the value of $v_2 \doteq$ $\doteq 83\,\mathrm{km{\cdot}h^{-1}}$.

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Problem 4 ... changing the transmission lines 3 points

We have a transmission line with high voltage $U_0 = 110\,\mathrm{kV}$, and we would like to increase it to extra-high voltage $U_1 = 400\,\mathrm{kV}$. Assuming the resistance of the line is constant, how much will the power losses on the line change? We are interested in the power loss ratio P_1/P_0.

Karel thought about changing the transmission lines.

We know that Ohm's law for a circuit or part of it says $U = RI$. Moreover, we can calculate the electric power P as $P = UI$. If we modify the formula by substituting the current from Ohm's law, we get

$$P = \frac{U^2}{R} \,.$$

So far, we have written the equation in general. Now, let us add the indices 0 and 1. Since the resistance remains constant, we will leave it without indices. Putting the indexed relations into the ratio, we easily get the result

$$\frac{P_1}{P_0} = \frac{\frac{U_1^2}{R}}{\frac{U_0^2}{R}} = \frac{U_1^2}{U_0^2} \doteq 13.2 \,.$$

Losses increase to 13.2 times the original value. In reality, the losses would probably increase more because with more power dissipation, the conductor temperature would stabilize at a higher temperature at which the conductor would have more resistance.

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[1] This stems from the fact that we can express the ratio of the hypotenuses in both triangles using the trigonometric function tangent.

Problem 5 ... swimming problem 3 points

Verča would like to go swimming, but she dislikes going into the water without eating. She has an average density of $945\,\mathrm{kg{\cdot}m^{-3}}$ *when she is hungry, and it is hard for her to dive. How many kilograms of food does Verča need to eat to have an average density of at least* $980\,\mathrm{kg{\cdot}m^{-3}}$*? Assume that she does not change her volume when she eats. The length of the pool is* $30\,\mathrm{m}$ *and its depth is* $3.1\,\mathrm{m}$*. Verča's weight before the meal is* $47\,\mathrm{kg}$*.*

Verča sometimes feels very empty inside.

First, we express the volume of Verča V. To do this, we use the information that before eating, her average density is ρ_0 and her mass is m_0

$$V = \frac{m_0}{\rho_0}\,.$$

Her volume does not change, while her average density must increase, hence

$$\frac{m_0}{\rho_0} = \frac{m_0 + \Delta m}{\rho_1}\,.$$

After simple algebraic manipulation, we get

$$\Delta m = m_0\left(\frac{\rho_1}{\rho_0} - 1\right) = 1.7\,\mathrm{kg}\,.$$

Thus, Verča has to eat 1.7 kg of food. The parameters of the pool were not needed to solve the problem.

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Problem 6 ... slide 3 points

The management of the Dormitories and Refectories decided to spend money meaningfully, so they built a slide from the roof of the "building A" of the 17th November dormitories directly to the door of the MFF's Impakt pavilion. The two buildings are $430\,\mathrm{m}$ *apart as the crow flies, and their height difference is* $59\,\mathrm{m}$*. What is the coefficient of friction of the slide if a student weighing* $60\,\mathrm{kg}$ *has thrown himself down it with a velocity* $v_0 = 8.5\,\mathrm{m{\cdot}s^{-1}}$ *and just stops at the end of it? Assume that the slide is an inclined plane.* *Eliška was late for a lecture.*

For the resultant force acting on the student on the slide applies $F = F_{\|} - F_t$, where $F_{\|}$ is the force that accelerates the student down the slide and is parallel to the plane of the slide, and conversely F_t is the frictional force that acts against the force $F_{\|}$. Moreover, the resultant force is constant, and thus, the acceleration must also be constant.

The student is also subject to a component of the gravitational force $F_{\perp}$ in a direction perpendicular to the slide surface, which is compensated by the reaction of the slide R (also perpendicular to its plane). The forces $F_{\perp}$ and $F_{\|}$ are components of the gravitational force F_g. There is an angle α between F_g and $F_{\|}$, which also corresponds to the angle of inclination of the slide, and we calculate it as $\alpha = \arctan(59\,\mathrm{m}/(430\,\mathrm{m}))$.

Because the student started running, he began to slip at v_0. He stopped with $v_1 = 0\,\mathrm{ms^{-1}}$ at the end of the slide. For the acceleration, we have the relation

$$a = \frac{v_1 - v_0}{t} = \frac{-v_0}{t}\,.$$

The distance of uniformly accelerated motion is the same as that of uniformly decelerated motion, allowing us to calculate it as

$$s = \frac{1}{2}at^2 = \frac{1}{2}v_0 t\,,$$

and if we express the time, we get

$$t = \frac{2s}{v_0}\,.$$

The distance s can be found from the knowledge of the horizontal and vertical dimensions of the slide as $s = l/\cos\alpha$, where $l = 430\,\mathrm{m}$ is the distance from the dormitories to the CUNI MFF's Impakt building.

After substituting in the equation for acceleration, we get

$$a = \frac{-v_0}{\frac{2s}{v_0}} = -\frac{v_0^2}{2s}\,.$$

We obtain $F_\perp = F_g\cos\alpha$ for the normal force, and $F_t = fF_\perp = fF_g\cos\alpha$ for the friction force. Finally, we use $F_\parallel = F_g\sin\alpha$ and $F = ma$. We get

$$F = F_\parallel - F_t\,,$$
$$ma = mg\sin\alpha - fmg\cos\alpha\,.$$

After some manipulation (we can notice that mass is irrelevant), we get

$$f = \frac{v^2}{2gl} + \tan\alpha\,,$$

and after inserting the numeric values, we obtain $f = 0.15$. Such a low coefficient is the result of the fact that the slope of the slide is very small. Next time, the Dormitories and Refectories management might choose a brachistochrone-shaped slide.

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Problem 7 ... Jarda's problems 3 points

When selecting problems for Physics Brawl Online, Jirka calculated that 20 % of the available problem assignments comes from Jarda. Yet, Jarda is the author of an incredible third of all the problems selected for the competition. Let us assume that FYKOS organizers do read the problem proposals when selecting them for the competition. How much more likely is it that the organizers will select a problem if the author is Jarda than if the proposal comes from another (average) organizer?

Jirka's original version of the problem did not pass the political censorship.

There are several ways of solving the problem. The first is by reasoning. Let's denote the total number of proposed problems by N and the number of tasks selected by n. We know that Jarda has proposed $0.2 \cdot N$ problems and $\frac{n}{3}$ were selected. We are interested in the quality of the tasks Jarda proposes, i.e., the ratio between the number of selected tasks and the number of tasks he proposed; which is equal to $n/(3 \cdot 0.2 \cdot N)$.

We now compare this ratio with the ratio for any other organizer. The other organizers proposed $0.8 \cdot N$ problems, from which a total of $\frac{2n}{3}$ were selected. Then the ratio is

$$\frac{\frac{n}{3}}{0.2 \cdot N} : \frac{\frac{2n}{3}}{0.8 \cdot N} = \frac{\frac{1}{3}}{\frac{2}{3}} \cdot \frac{0.8}{0.2} = 2$$

Thus, we discovered that Jarda's problems are approximately twice as successful as those of the other organizers.

Alternatively, we could solve the problem using conditional probability. Our goal is to calculate the probability that the problem is chosen given that it comes from Jarda (we will denote it by $P(\text{chosen}\,|\,\text{Jarda})$; the symbol $|$ denotes "under the condition"), and compare it with the probability of being selected, given that it comes from any other organizer.

For conditional probability

$$P(\text{selected}, |\, \text{Jarda}) = \frac{P(\text{selected} \cap \text{Jarda})}{P(\text{Jarda})}\,,$$

where the probability that the selected problem is Jarda's, i.e., $P(\text{selected} \cap \text{Jarda})$, is given again via conditional probability, this time using the probability that Jarda is the author of the selected task – so $P(\text{Jarda}|\text{selected})$. We know that this probability is equal to $1/3$.

In total, we have

$$P(\text{selected}\,|\,\text{Jarda}) = P(\text{Jarda}\,|\,\text{selected}) \cdot \frac{P(\text{selected})}{P(\text{Jarda})}\,,$$

while the probability that the task will be selected is unknown. Similarly, for any other organizer (denoted as "other") we have

$$P(\text{selected}\,|\,\text{other}) = P(\text{other}\,|\,\text{selected}) \cdot \frac{P(\text{selected})}{P(\text{other})}\,.$$

Now, we want to find out how much more likely Jarda's problems are to be selected, so we are interested in the proportion of conditional probabilities expressed. We get

$$\frac{P(\text{selected}\,|\,\text{Jarda})}{P(\text{selected}\,|\,\text{other})} = \frac{P(\text{Jarda}, |\,\text{selected})}{P(\text{other}\,|\,\text{selected})} \cdot \frac{P(\text{other})}{P(\text{Jarda})} = \frac{\frac{1}{3}}{\frac{2}{3}} \cdot \frac{0.8}{0.2} = 2\,.$$

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Problem 8 ... irresistibly attractive 4 points

Jindra can't find a girlfriend, so he orders a female-attracting device from a dubious online store. He received a pocket black hole. At a distance of $5\,\mathrm{m}$*, the black hole exerts gravitational acceleration of* $9.81\,\mathrm{m{\cdot}s^{-2}}$ *on all bodies (including girls). Calculate the Schwarzschild radius of this black hole.* *Jindra thought about complaining, but the black hole swallowed him.*

The well-known relation for the Schwarzschild radius of a black hole is

$$R_S = \frac{2GM}{c^2}\,, \tag{1}$$

where G is the gravitational constant, M is the mass of the black hole, and c is the speed of light. We assume the Schwarzschild radius of this black hole to be orders of magnitude smaller than 5 m. Therefore at a distance of $r = 5\,\mathrm{m}$, we can use the relation from classical physics

$$a = \frac{GM}{r^2}\,. \tag{2}$$

where $a = 9.81\,\mathrm{m{\cdot}s^{-2}}$ is the gravitational acceleration. We express the mass M from the equation (1) and plug it into the equation (2)

$$a = \frac{R_\mathrm{S} c^2}{2r^2}.$$

Now we express the Schwarzschild radius R_S and plug in the numbers

$$R_\mathrm{S} = \frac{2ar^2}{c^2} \doteq 5.5 \cdot 10^{-15}\,\mathrm{m}\,.$$

The pocket black hole has a Schwarzschild radius $R_\mathrm{S} \doteq 5.5 \cdot 10^{-15}\,\mathrm{m}$, so our initial assumption of a negligible black hole radius was confirmed. Our usage of the classical physics calculation for gravitational acceleration was justified.

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Problem 9 ... flying droplets 4 points

Rotating around a horizontal axis is a wheel with an outer diameter of $d = 65.8\,\mathrm{cm}$ upon which it is raining. Water droplets collide inelastically with the wheel's surface, but subsequently they can detach from it. What is the minimum angular rotation speed for water droplets to depart from the entire upper half of the wheel's perimeter?

Jindra rode his bike through the puddles.

Let's move into a system associated with the rotating wheel. Water droplets experience a gravitational acceleration $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$ downward and a centrifugal acceleration $\omega^2 r$ outward from the axis of rotation, where $r = 32.9\,\mathrm{cm}$ is the radius of the wheel, and ω is the angular velocity of rotation.

Water will depart from the wheel if the centrifugal force overcomes the radial component of the gravitational acceleration. We will measure the angle α from the vertical. The radial component of the gravitational acceleration is

$$g_r = g\cos\alpha,$$

where a positive sign indicates the direction inward towards the axis. A droplet located at position α on the wheel will depart if

$$g\cos\alpha < \omega^2 r,$$
$$\omega^2 > \frac{g\cos\alpha}{r}.$$

For this inequality to hold for all angles α from 0 to 2π it must be the case that

$$\omega > \sqrt{\frac{g}{r}}.$$

After substituting the given values, the result is $\omega > 5.46\,\mathrm{rad\cdot s^{-1}}$. The minimum angular rotation speed of the wheel for droplets to depart from the entire circumference is $5.46\,\mathrm{rad\cdot s^{-1}}$.

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Problem 10 ... stealing with a spring 4 points

We attach a spring with a stiffness $k = 5.2\,\mathrm{N\cdot m^{-1}}$ and a relaxed length $l_0 = 15\,\mathrm{cm}$ to a body with mass $m = 120\,\mathrm{g}$. We then start pulling at its other end at a constant speed $v = 65\,\mathrm{cm\cdot s^{-1}}$. To what maximum length does the spring stretch? The motion takes place on a smooth horizontal plane. *Jarda would like to become a pickpocket.*

Let's analyze the situation in the reference system of the hand. In this system, the body initially attains speed v, resulting in kinetic energy $mv^2/2$. That is all transformed into elastic potential energy $kx^2/2$ when the spring is the most stretched. This provides us with its maximum length

$$l = l_0 + v\sqrt{\frac{m}{k}} \doteq 25\,\mathrm{cm}\,.$$

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Problem 11 ... radiocarbon dating 5 points

Zuzka went to Abusir for archaeological excavations. In the tomb of an ancient Egyptian dignitary, she found a sample of wood, which she took to analyze on a mass spectrometer. The ratio of carbon isotopes in the sample was measured as $p_{^{14}\mathrm{C}/^{12}\mathrm{C}} = 9.22\cdot 10^{-13}$. In what year was the dignitary buried? Using the Gregorian calendar, write the years BCE with a minus sign. The half-life of ^{14}C is $T = 5\,730\,\mathrm{yr}$, and the ratio of isotopes in the atmosphere is historically constant $p_0 = 1.25\cdot 10^{-12}$. Assume that the tree was cut down shortly before the dignitary's burial. *Jindra came up with the origin of the problem only after Terka pointed it out to him.*

In the upper atmosphere, radioactive carbon atoms are naturally formed from nitrogen $^{14}\mathrm{N}$ by exposure to cosmic rays. Through photosynthesis, the radioactive carbon isotope is incorporated into plant cells and through the food chain into the bodies of animals. This fact establishes the same ratio p_0 of carbon isotopes $^{14}\mathrm{C}/^{12}\mathrm{C}$ in living organisms and the atmosphere.

When a living organism dies (e.g., when a tree is cut down), its carbon exchange with the environment stops. Thus, there is no replenishment of the decaying isotope $^{14}\mathrm{C}$, and the ratio of $^{14}\mathrm{C}/^{12}\mathrm{C}$ decreases exponentially with time, with a half-life of $T = 5\,730\,\mathrm{yr}$. We calculate the age of the tomb t from the equation

$$\frac{p_{^{14}\mathrm{C}/^{12}\mathrm{C}}}{p_0} = 2^{-\frac{t}{T}}\,,$$

$$t = -T\log_2\left(\frac{p_{^{14}\mathrm{C}/^{12}\mathrm{C}}}{p_0}\right)\,,$$

$$t = 2\,516\,\mathrm{yr}\,.$$

Subtracting the age of the wood from the current year of 2023, we get a burial year of 493 BCE, which we round and write in the solution as -490.

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Problem 12 ... curious cyclist

4 points

Kuba bought a new bicycle and wanted to find out the size of the rolling resistance coefficient ξ of his wheels. He noticed that the bike goes downhill on its own if the plane of the slope makes an angle with the horizontal greater than $\alpha = 0.5\,°$. The diameter of the bicycle wheels is $d = 67\,\mathrm{cm}$. Can you help Kuba? *Kuba likes cycling.*

If Kuba goes down a slope that has a deviation from the horizontal direction equal to α, the force in the direction of motion of the cyclist $\vec{F_1}$ and the rolling resistance force $\vec{F_v}$ will be in equilibrium. Thus, the bicycle will move in a uniform linear motion and

$$F_1 = F_v = F_n \frac{\xi}{d/2}\,.$$

The figure shows that $F_1 = F_G \sin(\alpha)$ and $F_n = F_G \cos(\alpha)$, so

$$F_G \sin(\alpha) = F_G \cos(\alpha) \frac{\xi}{d/2}\,.$$

After expressing ξ and substituting, we get the result

$$\xi = \frac{d}{2} \tan(\alpha) \doteq 2.9\,\mathrm{mm}.$$

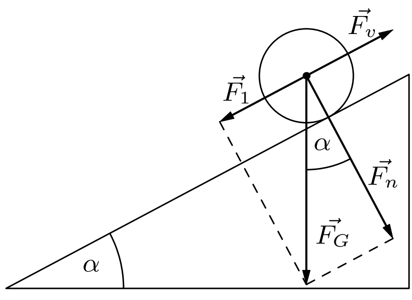

Figure 3: Decomposition of forces.

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Problem 13 ... carrying a box 5 points

As Lego carried the box, he was pondering about force he was applying to it. The box has a mass $m = 7.5\,\mathrm{kg}$ and is shaped like a rectangular cuboid. Lego holds it by pushing on two opposite (vertical) sides, and the coefficient of friction between the sides and Lego's hands is $f = 0.45$. What is the magnitude of the force exerted by one of Lego's hands on the box?

Lego was carrying a lot of things at the camp

The hands are pushing on the box from opposite sides, so they both have to push with the same normal force, let's call it F_N. Then the friction force between each of the hands and the box is $F_\mathrm{t} = fF_\mathrm{N}$. A gravity $F_g = mg$ acts on the box as well. In order for Lego to carry the box, the frictional forces between the box and the hands must compensate for this force. Thus

$$\begin{aligned} 2F_\mathrm{t} &= F_g \\ 2fF_\mathrm{N} &= mg \\ F_\mathrm{N} &= \frac{mg}{2f}\,. \end{aligned}$$

One might expect this to be the result, but it is not! The point is that the frictional force between the box and the hand is also a force that the hand exerts on the box. So each hand exerts forces F_N and F_t on the box, and these forces are perpendicular to each other, so if we want to know the magnitude of the total force exerted by the hand on the box, we get it as

$$|F| = \sqrt{F_\mathrm{N}^2 + F_\mathrm{t}^2} = F_\mathrm{N}\sqrt{1+f^2} = \frac{mg}{2f}\sqrt{1+f^2} = 90\,\mathrm{N}\,.$$

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Problem 14 ... divider No.1 4 points

Petr needed a $3\,\mathrm{V}$ voltage source, but he only had a $12\,\mathrm{V}$ source and resistors. So he decided to build a voltage divider from the schematic 4. What value of R did Peter have to choose to get $3\,\mathrm{V}$ at V_+?

I wanted to reminisce about electrical engineering.

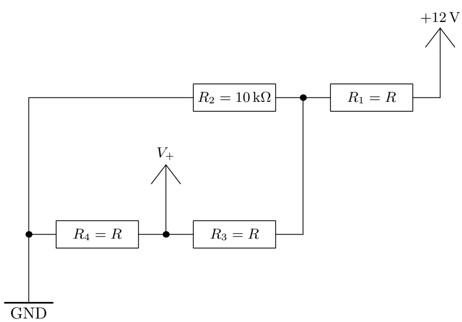

Figure 4: Circuit diagram

The problem's name reveals that we need a voltage divider (sometimes called a potential divider). It is a device consisting of two resistors connected in series, where we use the voltage on the second resistor (in the direction of the current) as a source. The formula for the voltage of an unloaded voltage divider on the second resistor in series is

$$U_{R_2} = U_{\mathrm{in}} \frac{R_2}{R_1 + R_2} \,,$$

where U_{in} is the voltage across the series circuit and U_{out} is the voltage across the second resistor. For the loaded divider, we have

$$U_{R_2} = U_{\mathrm{in}} \frac{R_2 R_{\mathrm{L}}}{R_1 R_2 + R_1 R_{\mathrm{L}} + R_2 R_{\mathrm{L}}} \,,$$

where R_{L} is the load resistance connected in parallel to R_2. First, let us consider that there are two resistive dividers, where the second one uses the output voltage of the first divider as its input voltage. However, the first divider is loaded by the second, and the schematic shows that the load resistance is $R_3 + R_4 = 2R$. Let us note that both resistances are the same in the second divider, and the previous formula for the unloaded divider shows that its output voltage will always be half of its input voltage. Thus, we only need to solve the circuit for the output voltage of the first divider. Then we apply the formula for the voltage of the loaded resistive divider, and we get

$$U_{\mathrm{out}} = U_{\mathrm{in}} \frac{R_2 \cdot 2R}{R_2 \cdot R + 2R^2 + R_2 \cdot 2R} \frac{1}{2} \,,$$

from this, we construct an equation for R

$$R = \frac{R_2 U_{\mathrm{in}} - 3 R_2 U_{\mathrm{out}}}{2 U_{\mathrm{out}}} \,.$$

After inserting the values from the assignment, we get

$$R = 5\,000\,\Omega \,.$$

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Problem 15 ... steel sphere floats 4 points

Karel found a steel sheet with a thickness of $\Delta r = 0.84\,\mathrm{mm}$ and was thinking about what to do with it. Since Karel likes to experiment, he made a hollow sphere of such a radius that when placed in the water, one half was above the surface and one below. Consider that the thickness of the sphere is just Δr and that the density of the steel is $\rho' = 7\,840\,\mathrm{kg{\cdot}m^{-3}}$. Find the outer radius of this sphere. *Karel was thinking on a boat.*

Let m be the sphere's mass, and ρ_{w} the density of water. Half of the volume of the sphere is immersed in water. In equilibrium, the buoyant force must be equal to the gravitational force, so we get the equation

$$mg = \frac{V}{2} \rho_{\mathrm{w}} g \,,$$

from which, after canceling g, we get

$$\frac{m}{V} = \frac{\rho_{\mathrm{w}}}{2} \,.$$

We express the sphere's density as $\rho = \frac{m}{V}$, where V is the total volume of the sphere and m is the mass of the sheet from which we made it. We determine the mass of the sheet by its density and the volume formed by the space between two spheres. The volume of the sheet will therefore be $V' = \frac{4}{3}\pi\left(r^3 - (r - \Delta r)^3\right)$, where r is the radius of the sphere. Substituting into the equation and rearranging, we get

$$\rho = \frac{m}{V} = \frac{\rho'\left(r^3 - (r - \Delta r)^3\right)}{r^3}\,,$$

while the weight of the air inside the sphere is equal to the buoyant force of the air acting on the part of the sphere above the water.

Substituting into the equation for the equality of forces, we get the equation of the third degree:

$$0 = \rho'\left(r^3 - (r - \Delta r)^3\right) - \frac{\rho_\mathrm{w}}{2}r^3\,,$$

which we solve numerically. The solutions will give one real root and two complex. The real root is $r = 0.0387\,\mathrm{m} = 3.87\,\mathrm{cm}$, which is the sought radius.

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Problem 16 ... modified solar system 5 points

Suppose the Sun had an effective temperature $T_2 = 8\,000\,\mathrm{K}$. By how many percent would the period of Jupiter's orbit have to be extended for the same amount of power to fall on it as it does now? *Danka was trying hard to come up with an interesting problem.*

According to the Stefan-Boltzmann law, the heat output of a star (an absolute black body) is

$$L = S_\odot \sigma T^4\,,$$

where $S_\odot$ is the surface area of the star, T is its temperature, and σ is the Stefan-Boltzmann constant.

The Sun radiates evenly throughout the space. The planet captures a fraction of this energy that is proportional to the cross-sectional area of the planet. So we want the following to hold

$$\pi r_\mathrm{J}^2 \frac{L_1}{4\pi d_1^2} = \pi r_\mathrm{J}^2 \frac{L_2}{4\pi d_2^2}\,,$$

where r_J is the radius of Jupiter and d_1 and d_2 are the distances from the Sun for surface temperatures T_1 and T_2, respectively. We substitute the heat output L from the first equation and get

$$d_2 = d_1 \left(\frac{T_2}{T_1}\right)^2\,.$$

Kepler's third law binds the orbital periods of the planets around the Sun

$$\frac{t_1^2}{d_1^3} = \frac{t_2^2}{d_2^3}\,,$$

from where we can already find the prolongation we are looking for as

$$p = \frac{t_2 - t_1}{t_1} = \left(\left(\frac{d_2}{d_1}\right)^{\frac{3}{2}} - 1\right) = \left(\left(\frac{T_2}{T_1}\right)^{3} - 1\right) = \left(\left(\left(\frac{4\pi R_{\odot}^2 \sigma}{L_1}\right)^{\frac{1}{4}} T_2\right)^{3} - 1\right) = 166\,\%\,.$$

Daniela Dupkalová
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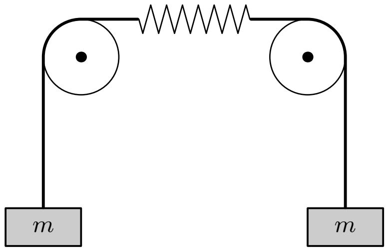
Problem 17 ... pulleys with elastic rope
4 points



Lego likes tasks with pulleys. But this time he wanted to come up with a problem that would model the fact that the rope has some elasticity. He decided to do this by splitting a perfectly stiff, weightless rope in the middle and then joining it with a weightless spring with stiffness $k = 78\,\mathrm{N{\cdot}m^{-1}}$. He then placed the modified rope on two fixed pulleys placed at the same height and hung a weight with mass $m = 9.0\,\mathrm{kg}$ on each end. He then held both the weights with just enough force so that the tension in the rope was $T = 12\,\mathrm{N}$. What is the acceleration with which the weights will move if they are released at the same time?

The problem came up to Lego's mind, when he had a lecture on pulleys at camp.

A quick way to solve this is to consider that a gravitational force mg will be applied to the block in the downward direction and a force from the rope in the upward direction. At the moment of letting go, the spring has not yet had time to stretch from the state it was in when we held the blocks. Thus, the force T will be applied to the rope, and hence the tension in the intangible rope must still be T. So the resulting force $mg - T$ is acting on both blocks and hence they will move with acceleration $a = g - T/m \doteq 8.5\,\mathrm{m{\cdot}s^{-2}}$.

But we will also describe a more complicated way (since it came to our mind earlier). The whole situation is symmetrical with respect to the centre of the spring, so it will not move. So we can imagine that this point is perfectly fixed to some wall. Thus we get a situation where a block of mass m oscillates on a half spring. That is effectively on a spring of stiffness $2k$. The angular frequency of the oscillation will be $\omega = \sqrt{2k/m}$.

The equilibrium position will be when this half of the spring is extended by $l_r = mg/2k$ compared to its rest length. We release the blocks when there is a tension T in the rope, then the spring half must be extended by $l_m = T/2k$. Since we are releasing the weight from rest, it will be at its maximum at the moment of release, and the magnitude of the oscillatory motion amplitude will therefore be $l_a = l_r - l_m = (mg - T)/2k$. At the same time, in addition to the position, there will be an acceleration in the amplitude, the magnitude of which will therefore be

$$a = l_a \omega^2 = \frac{mg - T}{2k}\frac{2k}{m} = g - \frac{T}{m} \doteq 8.5\,\mathrm{m{\cdot}s^{-2}}\,.$$

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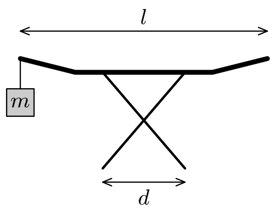
Problem 18 ... falling clothes horse
4 points



Matěj has a clothes horse with a width of $l = 180\,\mathrm{cm}$ on which he spreads his clothes evenly after washing. On the left edge of the dryer, he places a weight of $m = 3\,\mathrm{kg}$. How long after we hang up the clothes will the dryer tip over if its leg span is $d = 60\,\mathrm{cm}$? The mass of the wet clothes is $M_0 = 6\,\mathrm{kg}$; this will decrease to $M_1 = 2\,\mathrm{kg}$ when the clothes are completely dry. The weight of the dryer itself is $2\,\mathrm{kg}$. For simplicity, consider that the water evaporates at a constant rate and the clothes dry up in one day.

Matěj cannot dry his clothes.

Let us denote the mass of the dryer by M_s. We can calculate the position of the center of gravity of the entire system as the weighted average of the positions of the centers of the individual bodies. Thus, at the beginning of the drying process, the total center of gravity is located at

$$x_0 = \frac{m\frac{l}{2}}{m + M_0 + M_\mathrm{s}} = 24.5\,\mathrm{cm}$$

from the center of the dryer. When the laundry dries up, it moves to

$$x_1 = \frac{m\frac{l}{2}}{m + M_1 + M_\mathrm{s}} = 38.6\,\mathrm{cm}\,,$$

which is more than $x_\mathrm{max} = 30\,\mathrm{cm}$, and therefore the dryer must tip over at some point. From the relation above, we express the mass of the laundry and substitute x_max for the position of the center of gravity to obtain the minimum mass of the laundry that will not yet allow the clothes horse to tip over

$$M_\mathrm{min} = \frac{ml}{2x_\mathrm{max}} - m - M_\mathrm{s} = 4\,\mathrm{kg}\,.$$

If the laundry dries at a constant rate, it will dry to this critical mass in $\frac{M_\mathrm{min}-M_1}{M_0-M_1} = 1/2$ of a day, or 12 hours.

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Problem 19 ... acoustic speedometer

5 points

When the car is stationary, 80 raindrops per second hit its windshield. What velocity is it traveling at if the current frequency of impacts is $230\,\mathrm{s^{-1}}$? The windshield has an area S and is inclined at an angle of 33° with respect to the ground. Raindrops fall vertically to the ground at a velocity $4.5\,\mathrm{m{\cdot}s^{-1}}$. *Jarda was afraid of a speeding fine.*

Let's find the number of raindrops that fall on the glass in one second. Let us denote the velocity at which they fall as v and their volumetric density in the air as n. Then, f_1 raindrops fall on the stationary glass during one second

$$f_1 = nvS\cos\alpha\,.$$

However, the situation is a bit more complicated when the car is moving with velocity u. Now, the volume collected by the front windshield of the car per unit time is equal to

$$Q = uS\sin\alpha + vS\cos\alpha\,.$$

We see that an additional term has been introduced here. From the difference in frequencies, we obtain

$$f_2 - f_1 = nuS\sin\alpha\,,$$

From there, we can easily express the final velocity of the car

$$u = \frac{f_2 - f_1}{nS\sin\alpha} = v\frac{f_2 - f_1}{f_1\tan\alpha} = 47\,\mathrm{km{\cdot}h^{-1}}\,.$$

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Problem 20 ... the Synchrotron in Grenoble 5 points

At the ESRF synchrotron in Grenoble, electrons with an energy of 6.03 GeV *orbit along the track with a circumference of* 844.4 m, *generating a current of* 35.8 mA. *How many electrons are there in the entire synchrotron at one moment?*

Jarda participated in a diffraction experiment.

Since the rest mass of an electron is $E_0 = 511\,\mathrm{keV}$, it is negligible compared to their total energy of $E = 6.03\,\mathrm{GeV}$, so they move strongly relativistically. We can calculate their speed if we start from the equation:

$$E = \frac{E_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}\,,$$

where v is their speed and c is the speed of light. From this:

$$\frac{v}{c} = \sqrt{1 - \left(\frac{E_0}{E}\right)^2} = 0.999\,999\,996\,.$$

Furthermore, due to the precision of quantities in the task and the precision of the required result, we can solve the problem with approximation $v = c$.

The frequency of electron circulation in the synchrotron is $f = c/l$, where $l = 844.4\,\mathrm{m}$ is its circumference. The current produced by a single electron is $I_\mathrm{e} = ef$, where e is the elementary charge.

The total number of electrons is then:

$$N = \frac{I}{I_\mathrm{e}} = \frac{I}{ef} = \frac{Il}{ec} = 630 \cdot 10^9\,.$$

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Problem 21 ... maximal pétanque 6 points

Vojta played pétanque, but he was not good at it. Therefore, he got angry and threw the ball he held in his hand as far away as he could. At what angle must it be thrown with velocity $v = 11\,\mathrm{m\cdot s^{-1}}$ *to go as far as possible if the ball continues to roll after impact? When the ball hits the ground, only the vertical component of the velocity is absorbed, the ball does not slip, and the rolling resistance coefficient is* $c = 0.17$. *Assume that the ball falls on a nearby horizontal plateau that is at the same height as was the ball when it left Vojta's hand.*

Vojta misunderstood the rules of pétanque.

We can determine the range from the known relation

$$d_\mathrm{v} = \frac{v^2}{g}\sin 2\alpha\,,$$

where $0^\circ < \alpha < 90^\circ$ denotes the angle at which we launched the ball, which we are looking for. If all the vertical component of the velocity is absorbed, the kinetic energy of a sphere of mass m immediately after impact will be

$$E_\mathrm{k} = \frac{1}{2}m\left(v\cos\alpha\right)^2\,,$$

against which the rolling resistance will do the work. Thus, the condition for the ball to stop will be

$$E_\mathrm{k} = d_\mathrm{k} mgc \quad \Rightarrow \quad d_\mathrm{k} = \frac{1}{2cg} (v\cos\alpha)^2 \,,$$

and in total, the ball travels a distance

$$d = d_\mathrm{k} + d_\mathrm{v} = \frac{v^2}{g} \left(\frac{1}{2c} \cos^2\alpha + \sin 2\alpha \right) .$$

For this distance to be maximal, the following expression must be maximal

$$\frac{1}{2c} \cos^2\alpha + \sin 2\alpha \,.$$

So let's find the derivative with respect to α and set the result equal to zero

$$-\frac{1}{2c} 2\cos\alpha\sin\alpha + 2\cos 2\alpha = 0 \quad \Rightarrow \quad \tan 2\alpha = 4c \,,$$

from where we get the optimal angle as $17\,^\circ$.

Note that if we informally put $c = \infty$, we get the well-known result $45\,^\circ$ for the ball not rolling.

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Problem 22 ... taxidermy of a cylinder 5 points

Let's have a homogeneous cylinder. Around its axis, we cut a smaller cylinder out of it. The hollow and the smaller cylinder are then released down the inclined plane. What is the radius of the smaller cylinder if it starts with 20 % more acceleration than the rest of the larger cylinder? Provide the answer in multiples of the original radius.

Jarda wanted to state a problem without any number. It didn't work out.

When moving on an inclined plane, the law of conservation of energy applies to a rolling body of mass m in the form

$$mgh = \frac{1}{2} mv^2 + \frac{1}{2} J\omega^2 \,, \tag{3}$$

where g is the gravitational acceleration, h is the height by which the body has descended, v is the velocity attained by its center, J is the moment of inertia with respect to the axis of symmetry, and ω is the angular velocity of rotation. For a cylinder, $J = \frac{1}{2} mr^2$ holds, where r is its radius. If the body is not circular, $\omega r = v$ must hold.

On an inclined plane, we release two bodies of different mass and radius. For each of them, we calculate its acceleration. We denote the mass of the original cylinder M, its radius R, the mass of the small cylinder as m, and its radius as r.

For a small cylinder

$$\frac{1}{2} mv^2 + \frac{1}{2} J\omega^2 = \frac{1}{2} mv^2 + \frac{1}{2}\frac{1}{2} mr^2\omega^2 = \frac{1}{2} mv^2 + \frac{1}{4} mv^2 = \frac{3}{4} mv^2 \,. \tag{4}$$

The law of conservation of energy is therefore in the form

$$mgh_1 = \frac{3}{4} mv_1^2 \quad \Rightarrow \quad a_1 = \frac{2}{3} g \sin\alpha \,,$$

where the index 1 denotes the change in height, velocity, and acceleration of the cylinder, and α is the angle of inclination of the plane with respect to the horizontal direction. We came to the acceleration by the complete time derivative of the law of conservation of energy, since $\dot{h}_1 = v_1 \sin\alpha$ and $\dot{v_1^2}/2 = a_1 v_1$, with v_1 then reduced on both sides of the equation.

The mass of the large cylinder is $M - m$ and its moment of inertia with respect to the axis of symmetry is

$$\frac{1}{2}MR^2 - \frac{1}{2}mr^2 \,.$$

The right hand side of the equation 3 is thus

$$\frac{1}{2}\left(M-m\right)v^2 + \frac{1}{2}\left(\frac{1}{2}MR^2 - \frac{1}{2}mr^2\right)\frac{v^2}{R^2} = \frac{1}{2}\left(M-m\right)v^2 + \frac{1}{4}\left(MR^2 - mr^2\right)\frac{v^2}{R^2} \,. \tag{5}$$

By the same process as above we find the acceleration of the rest of the cylinder as

$$\left(M-m\right)gh_2 = \frac{1}{2}\left(M-m\right)v_2^2 + \frac{1}{4}\left(MR^2 - mr^2\right)\frac{v_2^2}{R^2} \,,$$

from which we get

$$a_2 = \frac{M-m}{\left(M-m\right) + \frac{1}{2}\left(M - m\frac{r^2}{R^2}\right)} g\sin\alpha \,.$$

From the condition $\frac{a_1}{a_2} = K = 1{,}2$ we get the equation

$$3M - 2m - m\frac{r^2}{R^2} = 3KM - 3Km \,.$$

We express m and M in terms of r and R, the cylinder length h and the density ρ as $m = \pi r^2 h\rho$ and $M = \pi R^2 h\rho$. Substituting into the previous equation and after subtracting π, h and ρ we get the biquadratic equation

$$r^4 + \left(2 - 3K\right)R^2 r^2 + 3\left(K-1\right)R^4 = 0 \,,$$

where the variable is r^2. The solution to this equation is

$$r^2 = \frac{-\left(2-3K\right) \pm \sqrt{\left(2-3K\right)^2 - 12\left(K-1\right)}}{2}R^2 = \frac{-2 + 3K \pm \left(4 - 3K\right)}{2}R^2 \,.$$

If we choose the $+$ sign, we get $r = R$ and independence on K, which makes no sense. So we choose the $-$ sign, which leads to

$$r = \sqrt{3\left(K-1\right)}R \,.$$

We see that it can never happen that the rest of the cylinder goes faster than the small cylinder. But at the same time, the problem has no solution for $K > \frac{4}{3}$ either, because then $r > R$ comes out. If $r \to R$, the rest of the cylinder becomes a hoop and achieves an acceleration of $\frac{1}{2}g\sin\alpha$. After setting $K = 1.2$, we get the result we are looking for

$$\frac{r}{R} = \sqrt{\frac{3}{5}} \doteq 0.775 \,.$$

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Problem 23 ... rats on rats — 4 points

Imagine a heap of rats, each of which has mass m. We arrange the rats in a 2D pyramid similar to Pascal's triangle. There will be one rat at the top, two rats below it, three in the next row, and so on. We have a lot of them, infinitely many. Each rat distributes its weight and the weight it carries to the rats below it. What is the total weight the legs of the rat on the far left must carry? Express the result as a multiple of m. If it would be infinite, enter 0. Consider the rats in a homogeneous gravitational field. *Karel was thinkig about Jára (da) Cimrman*

The main difficulty of the problem lies in the scope of the problem statement and in the understanding of the question itself. There are several possible approaches. A practical option is, for example, to make a spreadsheet in Excel and see what the values are close to; indeed, after a few rows, they start to settle around $2m$.

The alternative is to manually count the elements. The weight that the rat must bear in each successive row is obtained by always taking half of the previous rat's load and adding m, which represents the weight of the rat itself (we must not forget this, as the assignment asks for the total weight that the rat's legs must bear). For simplicity, let's consider only multiples of m – if the rat in the n-th row on the left carries a weight of m_n, we denote $a_n = m_n/m$. Let's write

$$a_1 = 1\,, \quad a_2 = \frac{a_1}{2} + 1 = \frac{3}{2}\,, \quad a_3 = \frac{7}{4}\,, \quad a_4 = \frac{15}{8}\,, \quad \dots$$

After a few more tries, it looks like we're still getting closer to 2 – in this way, this approach is similar to the previous one mentioned. For both, however, we don't know for sure if the value 2 is accurate, but you'll find that during the competition after entering it.

A better approach is to note what each member satisfies

$$a_n = 1 + \frac{a_{n-1}}{2} = 1 + \frac{1}{2} + \frac{a_{n-2}}{4} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{a_{n-3}}{8} = \dots = \sum_{i=0}^{n-1} \frac{1}{2^i}\,,$$

where in the notation on the right side we choose the summation index i from 0 to $n-1$ so that the sum corresponds to the actual situation. Now we can compute the limit as $n \to \infty$, respectively add up the infinite geometric series to get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2^i} = \frac{1}{1-\frac{1}{2}} = 2\,.$$

So we actually got the expected result of 2.

We will show one more, so far the most crafty method, that will also give us the correct result. Assuming we know the result, let's denote it x. If our sequence is indeed close to some real number, then it must be true that in the next row, the result is practically the same; so we construct the equation

$$x = 1 + \frac{x}{2} \qquad \frac{x}{2} = 1 \qquad x = 2\,.$$

Yet again we get the result that the multiple of the weight m carried by the bottom left (or bottom right) rat is 2. This approach is probably the fastest.

In all cases, the result is only valid for obedient rats that exert an even load on their mates below them, they all stand in a homogeneous gravitational field, and there is an infinite number of them. However poor rats in the middle have to carry an infinite load. *But the more rats, the more hatred towards them, as the classic, Jára Cimrman, wished in his pedagogical rules.*

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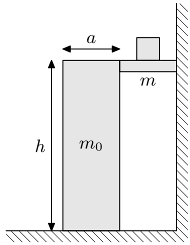
Problem 24 ... postmodern art 5 points

Lego wanted to create a sculpture. He therefore took a rectangular cuboid with mass $m_0 = 18.5\,\mathrm{kg}$, a square base with side length $a = 60.0\,\mathrm{cm}$ and height $h = 185\,\mathrm{cm}$, which he placed next to a solid wall (the sides of the cuboid are parallel to it). He then inserted a slab between the block (cuboid) and the wall so that it was held horizontally at height h only by friction. The coefficients of friction between the slab and the block, and the slab and the wall are $f = 0.42$. The coefficient of friction between the cuboid and the ground is effectively infinite. Lego, nevertheless, wants to put a weight on the middle of the slab. What is the largest sum of masses of the weights and the slab that the block can hold? The slab and the cuboid are homogeneous.



Lego's problems are evolving.

We are investigating when the block will not hold the slab anymore. We know that the friction between the cuboid and the ground is effectively infinite so that the cuboid does not slip (we can imagine a stopper behind the block that will not let it go any further). Thus, the forces acting on the block can always be in equilibrium. What, however, can cause a block to fail to hold the slab? It may be that the torques are not in equilibrium. It might seem that due to the unlimited friction between the block and the ground, the slab can push on the block with an arbitrary force. However, this is not true because the cuboid would topple over for some magnitude of force. It cannot exert such forces on the slab, so for cases where such a force would be needed to hold the slab, the slab would fall.

Thus, we are interested in the total sum of the torques of the forces acting on the block. We will choose the back edge on the ground as the axis of rotation since this is the edge around which the cuboid would start to overturn if the force exerted on it by the slab were too great.

What are all the forces acting on the block? It is gravity, the normal and frictional force between it and the slab, and the normal and frictional force between it and the ground. We will discuss these forces, and especially, their torque, in that order.

The block has mass m_0, so the weight of the block is $m_0 g$. Since the block is homogeneous, this force will act at its center, and hence, at a horizontal distance $a/2$ from the back edge. The torque with which this force acts on the cuboid will, therefore, be

$$M_g = \frac{1}{2} a m_0 g \,.$$

Let's examine the forces between the block and the slab. Let us denote the sum of the mass of the slab and the weight as m. The slab is pushed upwards by frictional forces only (between the slab, the wall, and the block). We can calculate the friction force as the friction coefficient f (f is the same for both surfaces) multiplied by the normal force that pushes the two surfaces together. Since the slab is not moving in the horizontal direction, the forces in that direction are balanced. Thus, the normal force on one side of the slab must be as large as that on the other. If we denote this force by F_N, the frictional force on each side will be $F_\mathrm{t} = f F_\mathrm{N}$. The sum of the frictional forces must compensate for the gravity of the slab and the weight, that is, $mg = 2F_\mathrm{t}$. From here, we can express the force pushing the block and the slab together as $F_\mathrm{N} = mg/(2f)$.

At what torques will these two forces act on the block? The slab will be pushing the block down with the frictional force, and this torque will, therefore, act in the same direction as the gravitational torque of the block itself. The horizontal distance of the center of this force from the axis of rotation is a, so the torque from the frictional force is

$$M_\mathrm{t} = aF_\mathrm{t} = \frac{1}{2}amg\,.$$

The torque from the normal force between the slab and the block will rotate the block in the opposite direction as the torque from the gravity and friction forces, so we assign the opposite sign to it. This force is horizontal, so its torque vector will be the same as the vertical distance of the origin and the axis of rotation, or h. Together, this torque will be

$$M_\mathrm{N} = -hF_\mathrm{N} = -\frac{1}{2f}hmg\,.$$

It is important to consider the effect between the block and the ground. The friction acts on the plane of the ground, which means that it will not exert any torque in the axis of rotation. However, it is crucial to understand the impact of the normal force. Normally, this force is distributed continuously over the entire contact area. In the absence of external forces, it can be assumed that the normal force will act uniformly at the center of the base.

When a force is applied to a block, its weight distribution changes. To better understand this concept, you can try standing upright while having a friend push you from the front. Even if you do not fall over, you'll notice that your weight shifts towards your heels. The same thing happens to the normal force distribution when we apply a force to the block.

Just as the frictional force is enough to keep the object from sliding, this normal force distribution is enough to keep the object from tipping over, unless the applied torque is too great.

The torque from the normal force from the ground will rotate the block in the same direction as the torque from the normal force from the slab. Thus, in the limit situation, when the normal force between the block and the slab is the maximum possible, the normal force between the ground and the block will act entirely in the axis of rotation (because it has zero torque there). If it did not operate entirely there and therefore had a non-zero torque, this would mean that it is possible to increase the force between the block and the slab, because a small enough increase would only tilt the block more backward. That is, when we are interested in the maximum possible weight of the slab and the weight, the torques of the forces between the block and the ground must be taken as zero in our chosen axis of rotation.

So, we get an identity that will hold in the limit case

$$\begin{aligned} M_g + M_\mathrm{t} + M_\mathrm{N} &= 0 \\ \frac{1}{2}am_0g + \frac{1}{2}amg - \frac{1}{2f}hmg &= 0 \\ m &= \frac{m_0}{\frac{h}{fa} - 1} = 2.92\,\mathrm{kg}\,. \end{aligned}$$

This is the maximum possible mass m that the block can support.

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Problem 25 ... ice in a cube

5 points

We enclose ice with a mass of 15 g *and temperature* 0 °C *under normal conditions in a hermetically sealed cubic container with a volume* 0.10 l. *Afterwards, we start heating it. What will be the pressure inside the cube when the temperature there reaches* 120 °C*?*

Jarda wanted to trick the participants, but he tricked himself instead.

Firstly, we will attempt the calculation while considering water an ideal gas, even though we might conclude that it was not the best idea.

The volume of air in the cube at the beginning is $V_\mathrm{v} = V - m/\rho_\mathrm{L} = 83.6\,\mathrm{cm^3}$, where $m = 15\,\mathrm{g}$ is the mass and $\rho_\mathrm{L} = 916.2\,\mathrm{kg{\cdot}m^{-3}}$ the density of ice at $0\,\mathrm{^\circ C}$. Because it is air under normal conditions, we can use the state equation to determine the amount of substance present in the cube as

$$n_\mathrm{v} = \frac{p_\mathrm{n} V_\mathrm{v}}{RT_\mathrm{n}} = 3.48\,\mathrm{mmol}\,,$$

where p_n and T_n are pressure and temperature under normal conditions.

At a temperature $T = 120\,\mathrm{^\circ C}$, all the water will have evaporated. The chemical amount of water is

$$n_{\mathrm{H_2O}} = \frac{m}{M_{\mathrm{H_2O}}} = 833\,\mathrm{mmol}\,.$$

The chemical amount of water is thus much higher; therefore, we can neglect the partial pressure of air. The total pressure will ultimately be

$$p = \frac{n_{\mathrm{H_2O}}}{V} RT = 27\,\mathrm{MPa}\,.$$

That is a very high pressure. However, water boils at higher temperatures under increased pressure. Therefore, not all the water will have evaporated even at 120 °C. Only a portion will have. Water will exert a partial pressure, which is the pressure of saturated water vapor at 120 °C, as at this point, no more water will evaporate. Its value is approximately $p_\mathrm{w} = 198.9\,\mathrm{kPa}$. It is possible to find this data on the internet, e.g., `https://www.engineeringtoolbox.com/water-vapor-saturation-pressure-d_599.html`.

To this pressure must be added the partial pressure of the air inside the cube. The volume of air is

$$V_\mathrm{v2} = V - V_\mathrm{w} = V - \frac{m}{\rho_{120\,\mathrm{^\circ C}}} = 84.1\,\mathrm{cm^3}\,.$$

In this calculation, we assumed that the mass of water that evaporated and is in the cube in gaseous form is negligible compared to the mass still in the liquid state. We observed that when all the water evaporates, its pressure is at least two orders of magnitude higher than the pressure of saturated water vapor. Therefore, the amount of evaporated water will also be two orders of magnitude lower than the mass of the remaining water. For this calculation, we used the density of water $\rho_{120\,\mathrm{^\circ C}} = 943\,\mathrm{kg{\cdot}m^{-3}}$ from source `https://www.engineeringtoolbox.com/water-density-specific-weight-d_595.html`.

For partial pressure of air

$$p_\mathrm{v} = n_\mathrm{v}\frac{RT}{V_\mathrm{v2}} = p_\mathrm{n}\frac{V_\mathrm{v}}{V_\mathrm{v2}}\frac{T}{T_\mathrm{n}} \doteq 135\,\mathrm{kPa}\,.$$

The total pressure inside the cube is

$$p_\mathrm{tot} = p_\mathrm{w} + p_\mathrm{v} = 334\,\mathrm{kPa} \doteq 330\,\mathrm{kPa}\,.$$

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Problem 26 ... When will I be there? 7 points

Jarda rides in an elevator that ascends at a constant speed. Since he is getting impatient, he is throwing his keys into the air. They always fly to the height of 62 cm. However, between one ejection and the subsequent catching of the keys, the elevator begins to brake steadily, so that the keys fly to a height of 72 cm and spend 0.15 s more time in the air. Determine the acceleration with which the elevator slowed down.

Jarda is in the elevator and already looking forward to bed.

When the elevator is braking on the way up, the inertial force in the elevator is acting upwards, so the acceleration of the keys will be smaller (we denote it as a). Since the height to which the keys flew is greater than the original height, braking occurs as the keys ascend.

When the elevator was moving steadily, the keys always spent in the air

$$h = \frac{1}{2}g\frac{t^2}{4} \quad\Rightarrow\quad t = 2\sqrt{\frac{2h}{g}} = 0.711\,\mathrm{s}\,.$$

We divide the time of the throw of the keys in the air when the elevator brakes into two – t_1 is the time from the ejection when the elevator is not yet braking and t_2 is the time the keys spend in the air when the acceleration a is applied to them. Thus $T = 2\sqrt{2h/g} + \Delta t = t_1 + t_2 = 0.861\,\mathrm{s}$. Thus for the initial velocity, we get $v_0 = \sqrt{2gh} = 3.49\,\mathrm{m{\cdot}s^{-1}}$.

Let's denote the velocity that the keys had at the moment of change of acceleration as v_1. Then

$$v_1 = \sqrt{v_0^2 - 2gh_1}\,,$$

where h_1 is the height at which they were at that moment. For time t_1 we then have

$$t_1 = \frac{v_0 - v_1}{g}\,.$$

Time t_2 is then composed of two parts – during the first part the keys were still flying up, and during the second they were falling down. We can write it as

$$t_2 = \frac{v_1}{a} + \sqrt{\frac{2H}{a}}\,.$$

Substituting into the equation for total time, we get

$$t_1 + t_2 = \frac{v_0}{g} + v_1 \frac{g-a}{ag} + \sqrt{\frac{2H}{a}} = T\,.$$

The last unknowns are v_1 and the variable a, which we want to find. From the law of conservation of energy, we know the velocity of the keys at the moment of change of acceleration. This is then converted entirely into a change in potential energy with the new acceleration in the lift, so we have an equation

$$v_0^2 = 2gh_1 + 2a\left(H - h_1\right) = 2\left(g - a\right)h_1 + 2aH \quad \Rightarrow \quad \frac{v_0^2 - 2aH}{2\left(g-a\right)} = h_1\,.$$

From here we substitute in the equation for v_1, which is

$$v_1 = \sqrt{a\frac{2gH - v_0^2}{g-a}}\,.$$

Substituting this into the equation for the times, we have an equation in which the only unknown is the acceleration. We have

$$\sqrt{\frac{g-a}{a}2g\left(H-h\right)} + g\sqrt{\frac{2H}{a}} = gT - v_0\,.$$

We could further modify the equation for a, but we will compute the value of a numerically. We get $a = 7.74\,\mathrm{m\cdot s^{-2}}$. The elevator therefore decelerated with acceleration $a_v = g - a \doteq 2.1\,\mathrm{m\cdot s^{-2}}$.

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Problem 27 ... insidious horn 5 points

At the FYKOS camp, participants measured a car's speed using the frequency shift of the horn. However, they encountered a problem – the horn's pitch changed between individual repetitions. Lego, therefore, came up with the following modification of the experiment: we measured the horn's frequency when the car was approaching us as $f_1 = 437\,\mathrm{Hz}$. Right after that, the frequency $f_2 = 415\,\mathrm{Hz}$ was measured when the car passed us closely. Assuming that neither the car's speed nor the frequency emitted by the horn has changed, what speed was the car traveling at? *Lego really came up with the idea at the camp during the presentations.*

The Doppler shift when the source approaches us is expressed by the relation

$$f_1 = f_0 \frac{v_c}{v_c - v}\,,$$

where f_1 is the frequency which we measure, f_0 is the frequency emitted by the source (in our case, the horn), $v_c = 343\,\mathrm{m\cdot s^{-1}}$ is the speed of sound in the air, and v is the speed of the source (in our case, the speed of the car).

When the source moves away from us, only the sign of the velocity changes, so the following holds

$$f_2 = f_0 \frac{v_c}{v_c + v}\,.$$

We get a system of two equations for the unknowns f_0 a v, where the task is to express v. We can solve the system, for example, by dividing the second equation by the first

$$\frac{f_2}{f_1} = \frac{v_c - v}{v_c + v}\,.$$

We multiply both sides by the denominator on the right side, move all terms containing v to one side, isolate v, and obtain the result as

$$v = v_c \frac{1 - \frac{f_2}{f_1}}{1 + \frac{f_2}{f_1}} = 31.9\,\mathrm{km{\cdot}h^{-1}}\,.$$

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Problem 28 ... incorrect voltage 5 points

A testing water electrolyzer produces $1.43\,\mathrm{g}$ *of hydrogen per hour. The device is powered by direct current from the source connected by cables, each having a resistance of* $3.1\,\mathrm{m\Omega}$. *Although the source exhibits an output voltage of* $1.95\,\mathrm{V}$, *this value is not directly associated with electrolysis. What voltage would be measured if we connected a voltmeter directly to the electrolyzer?* *Jarda produces hydrogen.*

Excluding the voltage necessary for water electrolysis, the source must supply voltage to overcome the ohmic resistance of supply cables. When measuring the voltage using the four-probe method, a lower value is obtained

$$U_\mathrm{e} = U - 2RI\,,$$

where I is the current passing through the entire device. The coefficient 2 must be included because one cable goes from the source to the electrolyzer and the second goes the other way. The amount of released hydrogen, according to Faraday's law of electrolysis, is proportional to the passing current

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{M_{\mathrm{H}_2}}{N_A}\frac{I}{2e}\,,$$

where the coefficient 2 accounts for the fact that two electrons are needed for the formation of every molecule H_2. The molar mass of hydrogen is $2.016\,\mathrm{g{\cdot}mol^{-1}}$. After obtaining I and substituting it into the first equation, the voltage under which the electrolysis occurs is

$$U_\mathrm{e} = U - 2R\frac{2eN_A}{M_{\mathrm{H}_2}}\frac{\mathrm{d}m}{\mathrm{d}t} \doteq 1.71\,\mathrm{V}\,.$$

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Problem 29 ... banded archerfish 6 points

Banded archerfish is a fish species that has found an original way to hunt for food. It approaches the surface and spits out a stream of water to knock down unsuspecting insects nearby. The insect falls into the water and has little time to escape. If the archerfish sees the insect sitting at an angle 35° relative to surface normal, how far must the insect sit from it to be knocked down? Consider that the fish can knock down insects at a maximum height of 3.0 m above the surface. *The fish splashed Jarda.*

From the last condition in the statement, we get that the speed at which the archerfish can spit water from its mouth is

$$v = \sqrt{2gh}\,,$$

where $h = 3.0\,\mathrm{m}$.

Furthermore, we need to use a protective parabola, whose equation for spraying from zero height is

$$y = -\frac{1}{4h}x^2 + h\,.$$

At the point where this parabola intersects with the direction towards the insect is the farthest position where the food can still be hit. Therefore, we need to determine this specific direction. It might seem to be the given 35° from the task, but the archerfish's eyes are below the water surface, so it is necessary to account for the refraction of light at the water-air interface. Using Snell's law for water with a refractive index of $n = 1.333$, we obtain

$$\beta = \arcsin(n \sin\alpha) = 50^\circ\,.$$

The line $y = x\cot\beta$ intersects with the protective parabola at points

$$0 = \frac{1}{4h}x^2 + x\cot\beta - h \quad \Rightarrow \quad x_{1,2} = 2h\frac{-\cos\beta \pm 1}{\sin\beta}\,,$$

where we are interested in the positive root of our solution. The total distance from the archerfish can, therefore, be at most

$$d = \frac{x_1}{\sin\beta} = 2h\frac{1-\cos\beta}{\sin^2\beta} = 2h\frac{1}{1+\cos\beta} \doteq 3.6\,\mathrm{m}\,.$$

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Problem 30 ... digging up 6 points

Inside a hollow planet, a special life form has evolved. The inhabitants of this vacuum bubble with a radius of $r = 1\,000\,\mathrm{km}$ decided to dig their way up to the planet's surface. Their scientists measured the density of the rock in several places and found that it decreases linearly with distance from the center of the planet. At the surface of their bubble, they measured a density of $9\,000\,\mathrm{kg{\cdot}m^{-3}}$ and $100\,\mathrm{km}$ further from the centre they measured $8\,800\,\mathrm{kg{\cdot}m^{-3}}$. This gave them an estimate that their planet was unlikely to have a radius larger than $5\,000\,\mathrm{km}$.

While digging the tunnel to the planet's surface, they decided to take a lunch break at a distance $R = 3\,000\,\mathrm{km}$ from the center of the planet. But they encountered a strange difficulty

– gravity. There's zero gravitational acceleration inside their bubble, so they were surprised to find something pulling them back down. Calculate the gravitational acceleration at the point of their lunch break. Assume that all material at distance x from the center of the planet has the same density. *Kuba was reading The Wandering Earth.*

We use Newton's shell theorem, which states that the gravitational field strength inside a spherical shell is zero. This is also the reason why there is zero gravitational acceleration inside this hollow planet. At the same time, this means that we can ignore all the mass that is above our explorers.

So we just need to determine how the mass in the sphere below the explorers acts gravitationally on them. In the center of this sphere is a vacuum bubble. Here we will use Gauss's law. It says that a spherical surface acts gravitationally on external objects as if all its mass is concentrated at its center. So we just need to determine the gravitational effect of a spherical surface with radius x and integrate that from r to R.

We know that the density of the planet ρ is some linear function depending on x. So it has the form $\rho(x) = ax + b$. From the measurements of scientists, we know that firstly $9\,000\,\mathrm{kg{\cdot}m^{-3}} = a \cdot 1\,000 \cdot 10^3\,\mathrm{m} + b$, and secondly $8\,800\,\mathrm{kg{\cdot}m^{-3}} = a \cdot 1\,100 \cdot 10^3\,\mathrm{m} + b$. After solving this system of equations, we get the result $a = -2 \cdot 10^{-3}\,\mathrm{kg{\cdot}m^{-4}}$ a $b = 11\,000\,\mathrm{kg{\cdot}m^{-3}}$. Thus, we get

$$\mathrm{d}g = \frac{G}{R^2}\,\mathrm{d}M = \frac{G}{R^2}\rho(x)\,\mathrm{d}V = \frac{G}{R^2}(ax+b)4\pi x^2\,\mathrm{d}x = \frac{4\pi G}{R^2}\left(ax^3 + bx^2\right)\mathrm{d}x\,.$$

We integrate this result from r to R

$$g = \frac{4\pi G}{R^2}\int_r^R \left(ax^3 + bx^2\right)\mathrm{d}x = \frac{4\pi G}{R^2}\left[\frac{a}{4}x^4 + \frac{b}{3}x^3\right]_r^R = \frac{\pi G}{3R^2}\left[3ax^4 + 4bx^3\right]_r^R\,.$$

After some manipulations and substitution, we then get

$$g = \frac{\pi G}{3R^2}\left[3a\left(R^4 - r^4\right) + 4b\left(R^3 - r^3\right)\right] \doteq 5.15\,\mathrm{m{\cdot}s^{-2}}\,.$$

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Problem 31 ... echooooooooooo — 6 points

In the middle of a long tunnel with radius $R = 15\,\mathrm{m}$ stands a point source of sound which emits a short beep. At a distance $D = 210\,\mathrm{m}$, also in the middle of the tunnel, we hear its intensity level as $60\,\mathrm{dB}$. How long after we initially hear the beep do we stop hearing its echo if each time it reflects off the wall its intensity decreases by $60\,\%$? The lowest sound intensity that can still be heard in the tunnel is $22\,\mathrm{dB}$.

In his younger days, Jarda often visited caves of the Moravian Karst.

The reduction of the sound intensity level takes place in two ways – by reflection, and by propagation in space, since the sound source is a point source. We express the sound intensity level as

$$L = 10\log\left(\frac{I}{I_0}\right)\,,$$

where I is the sound intensity at a given location and $I_0 = 10^{-12}\,\mathrm{W{\cdot}m^{-2}}$ is the intensity of the threshold of hearing.

A point source is characterized by the fact that we can neglect its dimensions with respect to the distance from it. Assume that the source isotropically radiates some power P. The intensity at distance r is thus

$$I = \frac{P}{4\pi r^2}\,.$$

Hence, the intensity decreases, not very surprisingly, with the square of the distance. We can easily calculate the acoustic power of the source from the problem statement and use it in further calculations.

We need to determine the number of reflections at which the sound intensity level will still be greater than $20\,\mathrm{dB}$, which corresponds to an intensity of $100I_0$. Sound is reflected off the walls, so it travels a greater distance to the point where we hear it. Since the situation is rotationally symmetric, we can only consider a 2D cross section. We mirror reflect the point at which we listen around the wall, several times. We then connect each of these points with a sound source and measure the distance. For the n-th reflected point, this will be

$$r_n = \sqrt{(2nR)^2 + D^2}\,.$$

We also note how many times the sound passes through the mirror walls, this represents the number of reflections, which is n. We then multiply the sound intensity calculated from the distance r_n by a factor of 0.4^n. We record the results in the table 1.

Table 1: Dependence of the sound intensity level on the number of reflections from the source.

$\frac{n}{1}$	$\frac{r_n}{\mathrm{m}}$	$\frac{I}{\mathrm{W{\cdot}m^{-2}}}$	$\frac{L}{\mathrm{dB}}$
0	210	$1.0 \cdot 10^{-6}$	60.0
1	212	$3.9 \cdot 10^{-7}$	55.9
2	218	$1.5 \cdot 10^{-7}$	51.7
3	228	$5.4 \cdot 10^{-8}$	47.3
4	242	$1.9 \cdot 10^{-8}$	42.9
5	258	$6.8 \cdot 10^{-9}$	38.3
6	277	$2.4 \cdot 10^{-9}$	33.7
7	297	$8.2 \cdot 10^{-10}$	29.1
8	319	$2.8 \cdot 10^{-10}$	24.5
9	342	$9.9 \cdot 10^{-11}$	19.9

We can see that after nine reflections the sound intensity level dropped below $22\,\mathrm{dB}$. This corresponds to a time delay

$$\Delta t = \frac{r_8 - r_0}{c} \doteq 0.32\,\mathrm{s}\,.$$

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Problem 32 ... maximal activity I

6 points

Jindra has $N_0 = 10^7$ atoms of the isotope ^{211}Bi. This isotope, with a half-life of $T_{\mathrm{Bi}} = 2.14\,\mathrm{min}$, transforms into the isotope ^{207}Tl, which then, with a half-life of $T_{\mathrm{Tl}} = 4.77\,\mathrm{min}$, transforms into the stable isotope ^{207}Pb. What is the maximal activity that the system can reach?

Jindra does not disdain any activity.

It might seem that to answer this question, we need to solve a system of differential equations describing the decay series, but this is not true. The maximal activity in the system will be at time $t = 0$, which we can also prove via simple reasoning.

The decay of a radioactive isotope can be described either by the half-life $T_{1/2}$ or by the decay constant λ. If there were N atoms of a given isotope in the system at time $t = 0$, then at time t there will only be

$$N(t) = N \cdot 2^{-\frac{t}{T_{1/2}}} = N\mathrm{e}^{-\lambda t} \tag{6}$$

atoms of that isotope, as some of them have already decayed. The relationship between the half-life and the decay constant is

$$\lambda = \frac{\ln 2}{T_{1/2}}\,.$$

The activity R of a radioactive isotope (the number of decays per second) depends on the number of atoms N in the system and the decay constant λ

$$R = \lambda N\,.$$

In our system, there are two radioactive isotopes ^{211}Bi and ^{207}Tl. Let's call the instantaneous number of bismuth atoms N_{Bi} and the instantaneous number of thallium atoms N_{Tl}. The activity depends on the instantaneous number of atoms of both isotopes as

$$R = \lambda_{\mathrm{Bi}} N_{\mathrm{Bi}} + \lambda_{\mathrm{Tl}} N_{\mathrm{Tl}}\,, \tag{7}$$

where λ_{Bi} and λ_{Tl} are the decay constants of the given isotopes. Given that $T_{\mathrm{Bi}} < T_{\mathrm{Tl}}$, it follows that $\lambda_{\mathrm{Bi}} > \lambda_{\mathrm{Tl}}$.

Now comes the key part of our reasoning. Because the decay constant of the bismuth isotope is greater than that of the thallium isotope, with the same number of atoms of both isotopes, the bismuth sample will exhibit higher activity. We start purely with bismuth atoms in the amount N_0 in our system. Over time, some of them decay into the thallium isotope. Further, some thallium atoms decay into the stable isotope of lead ^{207}Pb. If M atoms of bismuth have decayed, then there are $N_{\mathrm{Bi}} = N_0 - M$ atoms of bismuth and $N_{\mathrm{Tl}} \leq M$ atoms of thallium in the system. Looking again at equation (7), we see that some bismuth atoms have been replaced by thallium atoms. As mentioned earlier, the thallium isotope has a lower activity than the same amount of the bismuth isotope. But the number of bismuth atoms only decreases according to equation (6), and there can never be more thallium atoms in the system than the number of decayed bismuth atoms. Therefore, the maximal activity in the system occurred at time $t = 0$ and had a value of

$$R_{max} = \lambda_{\mathrm{Bi}} N_0 = 5.40 \cdot 10^4\,\mathrm{s}^{-1}\,.$$

The maximal activity in our system had a value of $54.0\,\mathrm{kBq}$.

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Problem 33 ... immersing

5 points

We have a negligibly thin hollow sphere and a hollow cube, both with volume $V = 5\,\mathrm{l}$ and full of air. What is the ratio of the work required to submerge them under water? The base of the cube is parallel to the surface (we don't rotate it during motion), both bodies are starting at the surface, and we want to get them to a state where they are completely submerged and touching the surface. As a result, sumbit a number greater than 1. *Matěj was in the bathtub.*

Solution via forces

To calculate the work, we will have to overcome the buoyant force – the gravitational force is in this case negligible and compensated by the buoyant force of the air (relative to $V \gg 0$). Its magnitude is always proportional to the volume V of the submerged part of the body according to Archimedes' law as $F = V\rho_\mathrm{v} g$, where ρ_v is the density of water and g is the gravitational acceleration.

The work required to submerge the cube is calculated straightforwardly as

$$W_\square = \int_0^a F\,\mathrm{d}h = \int_0^{\sqrt[3]{V}} g\rho_\mathrm{v} \sqrt[3]{V}^2 h\,\mathrm{d}h = \frac{1}{2} g\rho_\mathrm{v} \sqrt[3]{V}^4 .$$

For a sphere, the situation is a little more complicated – the submerged part is a spherical cap, and its volume is determined by the well-known relation

$$V_\mathrm{v} = \frac{\pi h^2}{3}(3r - h)\,,$$

where h is the height of the cap and $r = \sqrt[3]{3V/(4\pi)}$ is the radius of the sphere. We can then write

$$W_\bigcirc = \int_0^{2r} F\,\mathrm{d}h = \int_0^{2r} g\rho_\mathrm{v} \frac{\pi h^2}{3}(3r - h)\,\mathrm{d}h = g\rho_\mathrm{v} \frac{4\pi}{3} r^4 = \sqrt[3]{\frac{3}{4\pi}} g\rho_\mathrm{v} \sqrt[3]{V}^4 .$$

The work for the immersion of the sphere is therefore greater, and the ratio we are looking for is obtained as

$$\frac{W_\bigcirc}{W_\square} = \sqrt[3]{\frac{6}{\pi}} \doteq 1.24\,.$$

Solution without forces and without integrals

For simplicity, we will neglect the weight of the air, but we will return to this simplification over time. With this assumption, the work done in immersion is transfers only as the increase in the potential energy of the water. Thus, our question becomes what is the ratio of the increase in potential energy of water when we submerge a sphere versus when we submerge a cube.

The assignment does not say what the surface area is, so we will assume for simplicity that it is infinitely large. In that case, by submerging the object below the surface, we move the water that was previously in its place to the surface. The change in potential energy is $\Delta E_p = mg\Delta h$, where g is the gravitational constant and $m = V\rho$ is the mass of the displaced water, which is the same for both the cube and the sphere, so the ratio of the work done will be equal to Δh.

The Δh is equal to the change of height of the center of gravity of the displaced water. As we have already said, the water is lifted to the water surface, so the new position of the center of gravity will be at the surface. Since the water is homogeneous and both the sphere and the cube are symmetric objects, we can say that the original centers of gravity were located where the centers of the submerged objects are after submergence. That is, for the sphere r below the surface and for the cube $a/2$ below the surface. All this reasoning can be symbolically rewritten as

$$\frac{W_{\bigcirc}}{W_{\square}} = \frac{\Delta E_{p\bigcirc}}{\Delta E_{p\square}} = \frac{mg\Delta h_{\bigcirc}}{mg\Delta h_{\square}} = \frac{r}{a/2}\,,$$

where $r = \sqrt[3]{3V/(4\pi)}$ is the radius of the sphere and $a = \sqrt[3]{V}$ is the side of the cube. By utilizing these expressions and evaluating, we receive the result

$$\frac{W_{\bigcirc}}{W_{\square}} = \frac{\sqrt[3]{3V/(4\pi)}}{\sqrt[3]{V}/2} = \sqrt[3]{\frac{6}{\pi}} \doteq 1.24\,.$$

Finally, we return to the neglection of the weight of air, we can note that the decreases in the potential energy of air will be in the same ratio, so in fact, neglecting air does not affect the resulting ratio at all.

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Problem 34 ... frog on a water lily 6 points

Jarda spends so much time in the garden that he enjoys getting to know its inhabitants. However, a frog sitting on a water lily in the garden pond always gets scared and jumps away. If the water lily leaf has a mass of 43 g *and the mass of the frog is* 150 g*, how far can it jump if it jumps up to a distance of* 2.1 m *on solid ground? Assume that the water lily does not bend and moves only in a horizontal direction and that it moves freely.*

Jarda is woken up by frog croaking at home.

Consider that this frog is highly intelligent and jumps up from the ground at such an angle that it can jump as far as possible. It is a well-known fact that this angle is 45° and that the maximum distance that an object launched from the ground at speed v_0 reaches is

$$L = \frac{v_0^2}{g}\,,$$

from which we can find the speed of the frog right after the jump as $v_0 = \sqrt{gL}$.

Thus, with this speed, the frog is able to jump from a surface on which it is sitting. However, when the frog jumps from the water lily, by the law of conservation of momentum, its speed relative to the ground is smaller. Let the initial horizontal velocity of the frog in the reference frame with the water lily be $v_0 \cos\alpha$, where α is the angle at which the frog jumps. Then the law of conservation of momentum has the form

$$Mv_{\mathrm{h}} = mv_{\mathrm{l}}\,,$$

where v_h is the horizontal speed of the frog, M its mass and v_l is the speed of the water lily and m is its mass. In the water lily reference frame, the speed of the frog is $v_0 \cos\alpha = v_\mathrm{l} + v_\mathrm{h}$, from which

$$v_\mathrm{h} = \frac{v_0 \cos\alpha}{1 + \frac{M}{m}}\,.$$

So, at this speed, the frog is moving horizontally with respect to the water surface (with respect to the ground). The time the frog spends in the air is

$$t = 2\frac{v_0 \sin\alpha}{g}\,,$$

so that it can reach a distance of

$$l = v_\mathrm{h} t = \frac{v_0^2 2\cos\alpha\sin\alpha}{g\left(1 + \frac{M}{m}\right)}\,.$$

We got an interesting result. The best angle for the frog to jump relative to the water lily in its rest frame is again 45°. The maximum distance the frog can reach is

$$l = \frac{v_0^2}{g\left(1 + \frac{M}{m}\right)} = \frac{L}{1 + \frac{M}{m}} \doteq 0.47\,\mathrm{m}\,.$$

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Problem 35 ... relativistic star 6 points

What would be the radius of a star with the same mass as the Sun, but its redshift would be so large that the wavelength of the radiation coming from its surface would double for an observer at infinity? For simplicity, consider a non-rotating spherically symmetric star.

Karel was thinking about neutron stars and black holes.

The phenomenon described in the problem statement is called a gravitational redshift. For a spherically symmetric gravitational field, we describe it with the equation

$$\frac{\lambda_\infty}{\lambda_0} = \left(1 - \frac{r_\mathrm{S}}{R}\right)^{-1/2}\,,$$

where λ_∞/λ_0 is the ratio of the wavelengths of the radiation at infinity and the source (in this case equal to 2), R is the radius of the star, and r_S denotes its Schwarzschild radius. We determine the Schwarzschild radius from the equation

$$r_\mathrm{S} = \frac{2GM}{c^2}\,,$$

where M is the star's mass and G is the gravitational constant. By substituting and modifying the equation, we get

$$R = \frac{1}{1 - \left(\frac{\lambda_\infty}{\lambda_0}\right)^{-2}} \frac{2GM}{c^2} = \frac{4}{3} r_\mathrm{S} \doteq 3\,938\,\mathrm{m}\,.$$

This number implies that it will not yet be a black hole, but it is not far from it, and such a star would probably collapse into a black hole. Typical neutron stars with masses close to the mass of the Sun can have radii on the order of 10 km.

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Problem 36 ... billiard

6 points

The FYKOS-bird is at the pool table, which is 3.5 by 7 feet and has a hole for balls in each corner. There are white and green balls on the table, which are otherwise the same, fitting just right into the holes and bouncing off the walls and each other with perfect elasticity. The FYKOS-bird originally placed the green one on the longer axis of symmetry of the table. He nudged the white towards the green. The green crashed into one of the corner holes. However, after several impacts from the longer edges of the table, the white ball ended up in the same hole. How many positions on the table are there to place the green ball for this situation to play out? Ignore the rotation of the balls. *Jarda mostly hits the hole with white.*

First, we will show that the sphere flies off at right angles in an elastic impact. Let us denote the initial velocity vector of one of the spheres as $\mathbf{v}_0$, its velocity vector after the collision as $\mathbf{v}$, and the velocity vector of the other sphere as $\mathbf{u}$. Then, the law of conservation of momentum and the law of conservation of energy hold in the form

$$\mathbf{v}_0 = \mathbf{v} + \mathbf{u}, \qquad v_0^2 = v^2 + u^2 \,.$$

By squaring the first equation, we get

$$v_0^2 = v^2 + u^2 + 2\mathbf{v} \cdot \mathbf{u} \,,$$

which when compared with the law of conservation of energy, gives the condition $\mathbf{v} \cdot \mathbf{u} = 0$, or that indeed, the velocity vectors after the collision are perpendicular to each other and the spheres move along perpendicular trajectories.

The table is symmetrical according to two axes. In the beginning, we chose one hole to hit the balls into. Since we are placing the green ball on one of the axes of symmetry, two out of four solutions for the four holes will be the same. On the other hand, because of the symmetry, the solutions for two holes will be different, each of which will be on one side of the table along the longer edge. Thus, we must multiply the number of solutions found for one ball by two at the end.

In an elastic impact with a wall, the tangential component of the sphere's velocity is preserved because the force from the wall acts only perpendicularly. This force will change the perpendicular component of the velocity to the opposite since the sphere's energy is conserved. Therefore, we can think of the wall as a mirror and the sphere's trajectory as a ray. We can stretch this behind the wall and mirror the holes in the table and the other side of the wall. We can do this several times in a row, showing other such reflections on the same and opposite wall behind the first wall.

According to the previous paragraph, we used mirror reflections to project the hole into which the two balls are to fall (see figure). The trajectories of the balls form a right triangle with legs leading from the original location to the real and depicted hole. The point at which the collision occurs thus lies on Thales's circles, which always have centers at a distance of

$$d_n = \frac{2ns}{2}$$

from the hole hit along the shorter side of the pool table. Its length is $s = 3.5$ feet. The radius of the circles is then d_n. The distance of the point of collision from this wall is

$$y_n = \sqrt{d_n^2 - \left(d_n - \frac{s}{2}\right)^2} = s\sqrt{n - \frac{1}{4}}\,.$$

Since the ratio of the longer side $l = 7$ feet to the shorter side s is two, all possible values of y_n must be less than $2s$, which corresponds to

$$\sqrt{n - \frac{1}{4}} < 2\,,$$

and we find that the highest index that satisfies this is $n = 4$. For only one hole, just four such positions have the properties as in the statement. As we commented above, we must multiply this result by two for all remaining holes, quickly verifying that no position lies in the center of the table. The correct answer to the problem is therefore 8.

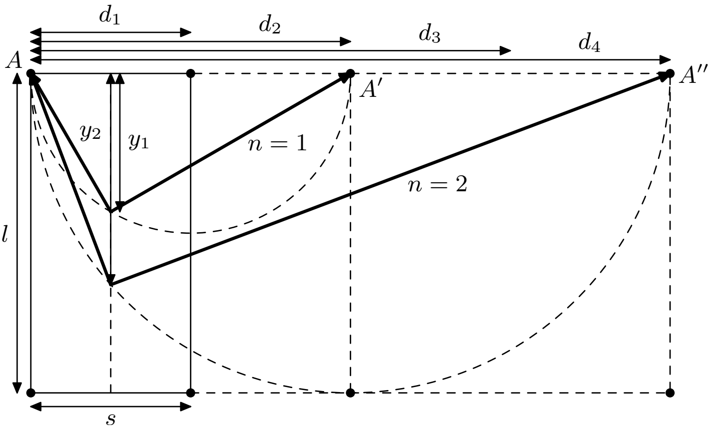

Figure 5: Zrcadlové zobrazení vybrané díry a situace pro $n = 1$ a $n = 2$.

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Problem 37 ... impedance spectroscopy — 7 points

In electrochemical experiments, we often encounter a method of sending an AC signal to the device, we then change its frequency and monitor the evolution of the impedance. Consider an alternative circuit, which consists of a parallel connection of a capacitor with capacitance C and a resistor with R_2, which are connected in series with R_1. Determine the largest possible phase shift by which the voltage is delayed behind the current if $4R_1 = R_2$.

Jarda examined data from his bachelor's thesis on electrolyzers.

The impedances of the individual elements are R_1, R_2 and $-i/\left(C\omega\right)$. The total impedance can therefore be written as

$$Z = R_1 + \left(\frac{1}{R_2} + iC\omega\right)^{-1} = R_1 + \frac{R_2}{1 + iR_2C\omega} = R_1 + \frac{R_2}{1 + R_2^2C^2\omega^2} - i\frac{R_2^2C\omega}{1 + R_2^2C^2\omega^2}\,.$$

If we plot a graph showing the imaginary component on the vertical axis and the real component of impedance as a function of frequency on the horizontal axis, we find that all points lie on a semicircle centered at $R_1 + R_2/2$ and with radius $R_2/2$. The entire semicircle lies below the real axis. Indeed, if we denote $\sin\psi = 2R_2C\omega/\left(1 + R_2^2C^2\omega^2\right)$, we get

$$\begin{aligned}
Z &= R_1 + \frac{R_2}{2} + \frac{R_2}{2}\exp(-i\psi) = R_1 + \frac{R_2}{2} + \frac{R_2}{2}\left(\cos(\psi) - i\sin(\psi)\right) = \\
&= R_1 + \frac{R_2}{2} + \frac{R_2}{2}\left(\frac{1 - R_2^2C^2\omega^2}{1 + R_2^2C^2\omega^2} - i\frac{2R_2C\omega}{1 + R_2^2C^2\omega^2}\right) = \\
&= R_1 + \frac{R_2}{2}\left(\frac{1 - R_2^2C^2\omega^2}{1 + R_2^2C^2\omega^2} + 1\right) - i\frac{R_2^2C\omega}{1 + R_2^2C^2\omega^2} = R_1 + \frac{R_2}{1 + R_2^2C^2\omega^2} - i\frac{R_2^2C\omega}{1 + R_2^2C^2\omega^2}\,.
\end{aligned}$$

The largest phase shift will be achieved if the absolute value of the ratio of the imaginary to the real component is the largest. Consider a straight line in the graph of the imaginary and real component that is gradually tilted counterclockwise from the vertical axis. This reduces the ratio of the imaginary to the real component. At one point, this line intersects the semicircle, which shows all possible impedances as a function of frequency. Thus, at this point the absolute value of the phase shift is the largest. At the same time, this straight line is now tangent to the circle. We get a right triangle whose hypotenuse is $R_1 + R_2/2$ and one of its legs is $R_2/2$. So the phase shift between voltage and current is finally

$$\varphi = -\arcsin\left(\frac{\frac{R_2}{2}}{R_1 + \frac{R_2}{2}}\right) = -\arcsin\left(\frac{R_2}{2R_1 + R_2}\right) = -\arcsin\left(\frac{2}{3}\right) \doteq -41.8\,^\circ\,.$$

The minus sign represents that the voltage is delayed behind the current, so the answer is the absolute value of the result.

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Problem 38 ... small hole in the water balloon — 7 points

We have a hollow sphere with thin walls of radius $R = 10.5\,\mathrm{cm}$. This sphere lies on (or is attached to at its lowest point) a horizontal plane; it is completely filled with water and has a hole at its highest point. Where do we need to drill another hole so that the water will spray out as far as possible from the point of contact of the sphere with the plane? We make the hole infinitely small. Enter the result as the angle formed by the line joining this hole and the center of the sphere with the vertical direction (i.e., a vector perpendicular to the plane and pointing away from it forms an angle of 0). *Lego thought, that a nice problem on hydro...*

We could use Bernoulli's law to get the rate at which the water will spray out of the hole, but we will simply use the reasoning from the law of energy conservation (from which Bernoulli himself is derived). As the water sprays, the water level will fall. Thus, the kinetic energy of the splashing water is gained at the expense of a decrease in the potential energy of the water. We can imagine that instead of water splashing out of the newly formed hole, which is there now, an element of water of mass m always teleports in from the top of the sphere, and it also flies out. We can do this reasoning because nothing happens to the water in all the other places in the container (which in turn is a consequence of the fact that we are considering a limitingly small hole; if it were not negligible, currents with non-negligible kinetic energy would be generated in the container). We have also subtly exploited that, thanks to the hole at the top of the vessel, the air pressure at the surface and in the new hole is the same. The splash element will thus have a kinetic energy corresponding to the decrease in potential energy due to the difference in the surface heights and the new hole. If we denote this difference by h, then

$$\frac{1}{2}mv_0^2 = mgh \to v_0 = \sqrt{2gh}\,.$$

So, we have the magnitude of the velocity at which the water is spraying. We still need the direction. This will be perpendicular to the wall at that point. This is because force is pressure times area, whereas pressure (in fluids) is a scalar quantity, so the direction of the force is given purely by the "direction of the area." Thus, the force pushing the water in the hole pushes it perpendicular to the surface of the hole.

Let us denote the angle between the line of the hole with the center of the sphere and the vertical direction as φ. Then the horizontal component of the initial velocity will be $v_{0x} = v_0 \sin\varphi$, and the vertical component will be $v_{0y} = v_0 \cos\varphi$.

The position of the hole relative to the point where the sphere touches the plane on which it lies can be expressed using the angle φ as $x_0 = R\sin\varphi$ and $y_0 = R(1+\cos\varphi)$. The top of the sphere is, of course, at height $2R$, so we can express the height difference between the hole and the top of the sphere h as $h = R(1-\cos\varphi)$.

Thus, in the vertical direction, the height above the plane will evolve as

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2\,.$$

Putting $y(t_\mathrm{d}) = 0$ gives the time for the water to fall. In general

$$t_\mathrm{d} = \frac{v_{0y} \pm \sqrt{v_{0y}^2 + 2gy_0}}{g}\,,$$

where we are only interested in the positive root, we don't yet account for the initial velocity and position, although we could (depending on preference).

The velocity in the horizontal direction does not change, so the distance from the point where the sphere touches the plane to the point where the water hits will be the product of the horizontal component of the velocity v_{0x} and the time to impact t_d, plus the initial horizontal distance x_0. Together, then, we get

$$x_\mathrm{d} = v_{0x}t_\mathrm{d} + x_0 = v_{0x}\frac{v_{0y} + \sqrt{v_{0y}^2 + 2gy_0}}{g} + x_0\,.$$

Insert for the initial positions and times

$$x_\mathrm{d} = \sqrt{2gh}\sin\varphi\frac{\sqrt{2gh}\cos\varphi + \sqrt{2gh\cos^2\varphi + 2gR(1+\cos\varphi)}}{g} + R\sin\varphi\,.$$

We can see that g is completely removed. Add h, and we get a relation depending only on φ

$$x_\mathrm{d} = \sqrt{2R(1-\cos\varphi)}\sin\varphi\left(\sqrt{2R(1-\cos\varphi)}\cos\varphi + \sqrt{2R(1-\cos\varphi)\cos^2\varphi + 2R(1+\cos\varphi)}\right) + R\sin\varphi\,.$$

We can take out R and get

$$x_\mathrm{d} = R\left(2\sqrt{1-\cos\varphi}\sin\varphi\left(\sqrt{1-\cos\varphi}\cos\varphi + \sqrt{(1-\cos\varphi)\cos^2\varphi + 1 + \cos\varphi}\right) + \sin\varphi\right)\,.$$

where R is a given constant, so we need to find φ that maximizes that bracket. Determining the maximum of such an expression analytically is difficult but probably completely impossible. So we'll plot the bracket into the graph 6.

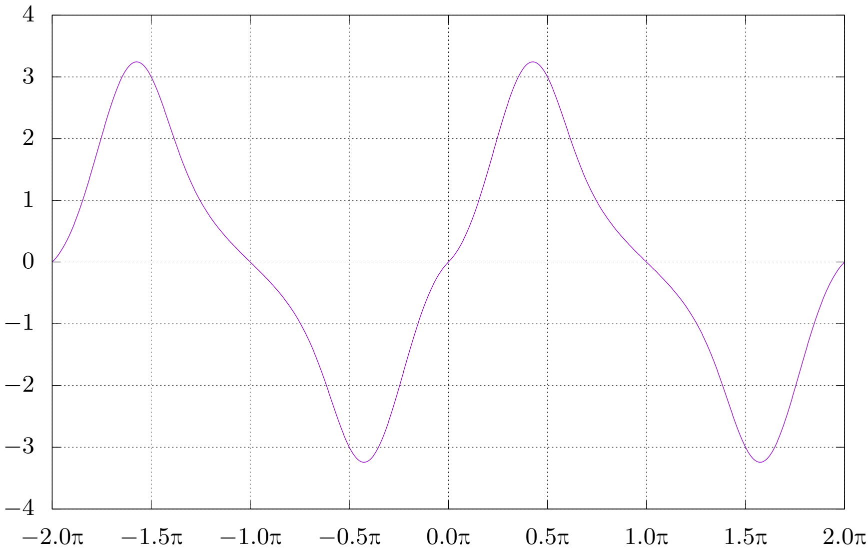

Figure 6: Distance to which the water x_d will penetrate, normalized to the radius of the sphere R depending on the angle of the hole φ.

The graph shows that the bracket takes a maximum for $\varphi = 1.34\,\mathrm{rad}$.

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Problem 39 . . . changing of our star 6 points

Consider a star similar to our sun that emitted most of its energy at a wavelength of $504.7\,\mathrm{nm}$ *and had a radius corresponding to* $0.990R_\odot$*, where* $R_\odot$ *is the radius of today's Sun. Over time, its composition has changed, increasing its radius to* $R_\odot$*, while shifting the wavelength* *of maximum emission by* 3.7 nm *towards the ultraviolet part of the spectrum. How much has the luminosity of this star increased during this time?*

Karel wondered about the history and future of the Sun.

For the luminosity of a black body

$$L = 4\pi R^2 \sigma T^4 \,,$$

where σ is the Stefan-Boltzmann constant. At the beginning, the star has radius R_0 and temperature T_0. After the transformation, its radius changes to $R_1 = R_\odot$ and its temperature to T. We calculate this temperature from the change in maximum wavelength. From Wien's displacement law we have

$$T_0 = \frac{b}{\lambda_0} \,, \qquad T = \frac{b}{\lambda_0 - \Delta\lambda} \,,$$

where $b = 2.898 \cdot 10^{-3}\,\mathrm{m \cdot K}$ is the *Wien constant* and $\Delta\lambda = 3.7\,\mathrm{nm}$ has a negative sign because its shift is to shorter wavelengths. For temperature, we have

$$T = \frac{\lambda_0}{\lambda_0 - \Delta\lambda} T_0 \,.$$

The ratio of luminosities is then equal to

$$\frac{L_1}{L_0} = \frac{R_1^2}{R_0^2} \cdot \left(\frac{\lambda_0}{\lambda_0 - \Delta\lambda}\right)^4 \,.$$

We are interested in how much the luminosity has changed, i.e., the number $L_1/L_0 - 1$ expressed as a percentage. We get

$$\frac{L_1}{L_0} - 1 = \frac{1}{0.99^2} \cdot \left(\frac{504.7\,\mathrm{nm}}{504.7\,\mathrm{nm} - 3.7\,\mathrm{nm}}\right)^4 - 1 = 0.0508 \,,$$

which is 5.08 %.

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Problem 40 ... top of the class 6 points

During one of the boring lessons at school, the students invented their own fun – they threw a small heavy object perpendicularly upwards so that it would be as close to the ceiling as possible, but at the same time, would not touch it. The ceiling is at a height $H = 2.7\,\mathrm{m}$ *from the point of the throw. However, when they did this fun in physics class, the teacher made them measure the initial velocity and its standard deviation as* $(7.0 \pm 0.5)\,\mathrm{m{\cdot}s^{-1}}$. *How likely are the students to hit the ceiling if the distribution of initial velocities is Gaussian?*

Jarda was listening to the Vašek's stories.

The velocity required to hit the ceiling is

$$u = \sqrt{2gH} = 7.28\,\mathrm{m{\cdot}s^{-1}} \,.$$

Let $v_0 = 7.0\,\mathrm{m{\cdot}s^{-1}}$ denote the mean initial velocity and $\Delta v = 0.5\,\mathrm{m{\cdot}s^{-1}}$ the standard deviation of the previous quantity. Then, the Gaussian curve for the given values has the form of

$$f(v) = \frac{1}{\sqrt{2\pi}\Delta v}\exp\left(-\frac{(v-v_0)^2}{2\left(\Delta v\right)^2}\right)\,.$$

We can find the probability that the velocity of the throw is greater than v by integrating the Gaussian curve in the limits from u to ∞ as

$$p = \frac{1}{\sqrt{2\pi}\Delta v}\int_u^\infty \exp\left(-\frac{(v-v_0)^2}{2\left(\Delta v\right)^2}\right)\mathrm{d}v \;\; = 28.8\,\% \doteq 29\,\%\,,$$

We had to calculate the integral numerically (e.g., using WolframAlpha).

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Problem 41 ... pressure difference 6 points

Jindra has a solid cylindrical tube with a $H = 50.0\,\mathrm{cm}$ length and a radius of $r = 4.00\,\mathrm{cm}$. One end of the tube is airtight sealed with a lid. Then, he held the tube by the lid and began to submerge it perpendicularly beneath the sea level with the open end. What will be the maximum difference between the pressures acting on the top and bottom lids of the tube during submersion? Jindra submerged the tube to a depth allowed by the length of his arm, $l = 65.0\,\mathrm{cm}$. The density of water is $\rho = 1\,024\,\mathrm{kg{\cdot}m^{-3}}$. The air at sea level is under normal conditions. Both sea water and air have the same temperature.

Jindra didn't know how else to entertain himself on a holiday by the sea.

Let's denote the depth of the bottom edge of the tube as d. The water inside the tube will rise above the bottom edge and compress the air inside. Let's denote the water column height above the bottom of the tube as h. Equilibrium occurs when the water pressure $p_\mathrm{a} + \rho g(d-h)$ balances the pressure of the compressed air inside the cylinder. The gravitational acceleration is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$, and the atmospheric pressure is $p_\mathrm{a} = 101\,325\,\mathrm{Pa}$ (see constants). Due to the slow submersion of the tube beneath the surface, air compression occurs isothermally. The air inside has the same temperature as the atmosphere and the sea.

As long as the tube lid is above the surface, atmospheric pressure p_a is applied to the top of the lid. The air inside the tube, compressed at a pressure $p_\mathrm{a} + \rho g(d-h)$, pushes on its bottom side. The difference in pressure acting on the lid is $\rho g(d-h)$.

The quantity $d-h$ is the depth of the water surface in the tube below sea level. If we sink the tube deeper, we increase d, the height of the water column in the tube will also increase h, but less than the increase in d. If the water level of the column remained at the same depth $d-h$ below sea level as before, the water pressure would remain the same, but the air volume would be less – the air would push the water level of the column down. If the height of the water column h remained the same when submerged, the air would have the same pressure as before, but the depth $d-h$ of the column would be greater than before – the water pressure would push the air in the tube.

Therefore, the depth $d-h$ increases continuously as the tube is submerged. Thus, the pressure difference $\rho g(d-h)$ increases until the lid is submerged at sea level.

The pressure applied from above changes once the lid is submerged below sea level. Now, the pressure of the water column above the lid and the atmosphere $p_\mathrm{a} + \rho g(d - H)$ is acting on the lid from above. From below, the pressure of the compressed air $p_\mathrm{a} + \rho g(d - h)$ is applied. The pressure difference is $\rho g(H - h)$. As we sink the tube deeper, the air is compressed under increasing water pressure. As the water column height inside the h increases, the pressure difference across the lid $\rho g(H - h)$ decreases.

The maximum pressure difference, therefore, occurs when the tube lid is aligned with the sea surface. The isothermal compression of the air inside the tube is described by equation:

$$p_1 V_1 = p_2 V_2 \,,$$

where p_1, p_2 are the initial and final pressures and V_1, V_2 are the initial and final gas volumes. The initial air pressure in the tube before plunging is $p_1 = p_\mathrm{a}$ and the initial volume is $V_1 = \pi r^2 H$. The final volume of air is $V_2 = \pi r^2 (H - h)$ and the final pressure is

$$p_2 = \frac{V_1}{V_2} p_1 = \frac{\pi r^2 H}{\pi r^2 (H - h)} p_\mathrm{a} = \frac{H}{H - h} p_\mathrm{a} \,.$$

The pressure of the water in the tube when the lid is at sea level $d = H$ is

$$p_2 = \rho g(H - h) + p_\mathrm{a} \,.$$

These two pressures must be equal, so we get the equation for h

$$\frac{H}{H - h} p_\mathrm{a} = \rho g(H - h) + p_\mathrm{a} \qquad \Rightarrow \qquad h^2 - \left(\frac{p_\mathrm{a}}{\rho g} + 2H\right) h + H^2 = 0 \,,$$

thus

$$h_{1,2} = \frac{1}{2}\left[\frac{p_\mathrm{a}}{\rho g} + 2H \pm \sqrt{\left(\frac{p_\mathrm{a}}{\rho g} + 2H\right)^2 - 4H^2}\right] = \frac{p_\mathrm{a}}{2\rho g} + H \pm \sqrt{\left(\frac{p_\mathrm{a}}{2\rho g}\right)^2 + \frac{p_\mathrm{a} H}{\rho g}} \,.$$

The height of the water column h must be less than the height of the tube H, so we are only interested in the root of the quadratic equation with the minus sign

$$h = \frac{p_\mathrm{a}}{2\rho g} + H - \sqrt{\left(\frac{p_\mathrm{a}}{2\rho g}\right)^2 + \frac{p_\mathrm{a} H}{\rho g}} \,.$$

This is plugged into the equation for the pressure difference acting on the lid

$$\Delta p = \rho g(d - h) = \rho g(H - h) = \rho g \sqrt{\left(\frac{p_\mathrm{a}}{2\rho g}\right)^2 + \frac{p_\mathrm{a} H}{\rho g}} - \frac{p_\mathrm{a}}{2} \,,$$

which gives the highest pressure difference $\Delta p = 4.80\,\mathrm{kPa}$.

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Problem 42 ... fireworks

6 points

After the competition, the FYKOS organizers planned a fireworks display. They will launch their firework at $45\,\mathrm{m\cdot s^{-1}}$ perpendicularly upwards, which will disintegrate into many small pieces in $3.3\,\mathrm{s}$. These fly off in all directions from the point of disintegration at $15\,\mathrm{m\cdot s^{-1}}$ relative to the reference frame of the original firework and glow for $5.5\,\mathrm{s}$. Determine the volume of space into which the fragments have managed to spread when they are extinguished.

Jarda likes to toast on fireworks.

The velocity of the reference frame of the firework at the moment of disintegration is $u = v_0 - gT = 12.63\,\mathrm{m\cdot s^{-1}}$ in the upward direction, where $T = 3.3\,\mathrm{s}$ is the time of the disintegration of the firework since launch and $v_0 = 45\,\mathrm{m\cdot s^{-1}}$ is the initial velocity. The problem is rotationally symmetric, so let us consider a cut through only one plane in which we introduce coordinates y upward and x to one of the sides so that the explosion occurred just on the $x = 0$ axis. Next, we will investigate the individual fragments into which the firework breaks up. Let us denote by α the angle of the fragments with respect to the ground, $v = 15\,\mathrm{m\cdot s^{-1}}$ the velocity of the fragments after the disintegration of the firework, $\tau = 5.5\,\mathrm{s}$ the time of their glow. The fragment then moves in the horizontal direction with velocity $v_x = v\cos\alpha$, while in the vertical direction its velocity varies due to gravitational acceleration as $v_y = u + v\sin\alpha - gt$, where t is the time since the disintegration of the firework. Integrating the two velocities with respect to time gives the dependence of position on time as

$$x = v\cos\alpha t,\; y = H + ut + v\sin\alpha t - \frac{1}{2}gt^2\,,$$

using $H = v_0T - 1/2\cdot gT^2 = 95.1\,\mathrm{m}$ as the disintegration height of the original firework.

We can notice that these coordinates form a circle with centre at $H + ut - 1/2\cdot gt^2$ and radius vt depending on the angle α. Now we need to find out if all the fragments are still in the air when the light goes out, or if some have already hit the ground. Plugging τ into the equation for the y coordinate and putting $y = 0$ gives the condition

$$0 = H + u\tau + v\sin\alpha_\mathrm{d}\tau - \frac{1}{2}g\tau^2 \quad\Rightarrow\quad \sin\alpha_\mathrm{d} = \frac{\frac{1}{2}g\tau^2 - H - u\tau}{v\tau} = 0.196\,.$$

The fragments, whose original angle was thus less than α_d, hit the ground while the others are still in the air. The shape whose volume we are now seeking is thus a spherical canopy. We calculate the volume of the spherical canopy as $V = \pi h^2\left(3r - h\right)/3$, where r is the radius of the sphere and h is the height from the cut-off wall. In our case, h is the height above the ground of the fragments that flew perpendicularly upwards in time τ after the disintegration of the firework, i.e.

$$h = H + \left(u + v\right)\tau - \frac{1}{2}g\tau^2 = 98.7\,\mathrm{m}\,,$$

while the radius is $r = v\tau = 82.5\,\mathrm{m}$. The search volume is thus

$$V = \frac{\pi\left(H + \left(u+v\right)\tau - \frac{1}{2}g\tau^2\right)^2}{3}\left(2v\tau - H - u\tau + \frac{1}{2}g\tau^2\right)\,,$$

$$V = \frac{\pi\left(v_0\left(T+\tau\right) - \frac{1}{2}g\left(T+\tau\right)^2 + v\tau\right)^2}{3}\left(2v\tau - v_0\left(T+\tau\right) + \frac{1}{2}g\left(T+\tau\right)^2\right)\,,$$

which is approximately $V \doteq 1.52 \cdot 10^6\,\mathrm{m^3}$.

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Problem 43 ... electron collision 6 points

In one accelerator, colliding electrons are flying towards each other, each with an energy of 104.5 GeV*. In a second accelerator, electrons with an energy of* 209 GeV *are flying towards a target made of stationary electrons. How many times more energy is available for the creation of matter in the first accelerator compared to the second?*

Jindra felt like colliding his electrons.

The electrons in both accelerators are highly relativistic $E \gg m_0c^2$, where $m_0 = 9.109 \cdot 10^{-31}\,\mathrm{kg} \doteq 511.0\,\mathrm{keV/c^2}$ is the rest mass of an electron, which we can find in the "overview of constants". The sum of the energies of the colliding particles is the same in both accelerators, but this does not determine the energy of the collision. In the second accelerator, the center of mass of the system of both electrons is also moving. Due to the conservation of momentum during the collision, the center of mass will move after the collision as well – thus, part of the kinetic energy is associated with the motion of the center of mass and cannot be used to create new particles. To find the energy available for the creation of new particles in the second accelerator, we must also study it in the coordinate system associated with the center of mass of the colliding electrons. In the first accelerator, the situation is simple – we are already in the system associated with the center of mass of the colliding electrons. Both electrons are identical, and both fly towards each other with the same kinetic energy, thus also with the same momentum and velocity. The energy available for the creation of new particles is the total energy of both electrons

$$E_1 = 2E_A = 209\,\mathrm{GeV}\,,$$

where we denoted E_A the energy of one electron in the first accelerator.

In the second accelerator, the moving electron has energy

$$E_B = \gamma m_0 c^2\,,$$

where $E_B = 209\,\mathrm{GeV}$ and γ is the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\,.$$

The Lorentz factor and the velocity of the electron are

$$\gamma = \frac{E_B}{m_0 c^2} = 4.090 \cdot 10^5\,,$$

$$v = c\sqrt{1 - \frac{1}{\gamma^2}} = c\sqrt{1 - \left(\frac{m_0 c^2}{E_B}\right)^2}\,.$$

The center of mass of the system moves with velocity

$$u = \frac{\gamma m_0 v}{\gamma m_0 + m_0} = \frac{\gamma}{\gamma + 1} v$$

towards the stationary electron. Now, we must relativistically transform the velocities of both electrons and determine their velocity relative to the center of mass. The velocity of the flying electron relative to the center of mass is

$$v_1 = \frac{v-u}{1-\frac{uv}{c^2}} = \frac{v - \frac{\gamma}{\gamma+1}v}{1-\frac{\gamma}{\gamma+1}\frac{v^2}{c^2}} = \frac{\frac{1}{\gamma+1}}{1-\frac{\gamma}{\gamma+1}\frac{v^2}{c^2}}v = \frac{1}{\gamma+1-\gamma\frac{v^2}{c^2}}v\,.$$

Now we will use the definition of γ and substitute it into the equation

$$v_1 = \frac{1}{1+\sqrt{1-\frac{v^2}{c^2}}}v = \frac{1}{1+\frac{1}{\gamma}}v = \frac{\gamma}{\gamma+1}v = u\,.$$

The stationary electron moves with the speed

$$v_2 = u = \frac{\gamma}{\gamma+1}v\,,$$

towards the center of mass. Therefore, the total energy available during the collision is

$$E_2 = 2\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}m_0c^2\,.$$

We simplify the fraction

$$\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{\gamma^2}{(\gamma+1)^2}\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{\gamma^2}{(\gamma+1)^2}\left(1-\frac{1}{\gamma^2}\right)}} = \frac{1}{\sqrt{1-\frac{\gamma-1}{\gamma+1}}} = \sqrt{\frac{\gamma+1}{2}}\,.$$

The total energy available during the collision is

$$E_2 = 2\sqrt{\frac{\gamma+1}{2}}m_0c^2 \doteq 462.2\,\mathrm{MeV}\,.$$

The ratio of the energies available for creating new particles in the first accelerator relative to the second is

$$\eta = \frac{E_1}{E_2} = 452.2 \doteq 452\,.$$

Although the sum of the kinetic energies of the electrons is the same in both accelerators, and thus the same work had to be done to accelerate them, the first accelerator offers 640 times more energy available for the creation of new particles. That's why modern large accelerators like the LHC are built to collide particles flying towards each other.

We would reach the same conclusion even if we summed the energies and momenta of all particles, subtracted their squares, and took the square root

$$E^2 = \left(\sum_i E_i\right)^2 - c^2\left(\left|\sum_i \mathbf{p}_i\right|\right)^2\,.$$

For the first accelerator, we get

$$E_1 = \sqrt{(2E_A)^2 - c^2\left(|0|\right)^2} = 2E_A = 209\,\mathrm{GeV}\,.$$

For the second accelerator, it turns out

$$E_2 = \sqrt{(E_B + m_0c^2)^2 - (\gamma m_0 vc)^2} = \sqrt{((\gamma+1)m_0c^2)^2 - \left(\gamma m_0c^2\sqrt{1-\frac{1}{\gamma^2}}\right)^2} =$$

$$= m_0c^2\sqrt{(\gamma+1)^2 - (\gamma^2-1)} = m_0c^2\sqrt{2\gamma+2} = 2\sqrt{\frac{\gamma+1}{2}}m_0c^2 \doteq 462.2\,\mathrm{MeV}\,.$$

We arrived at the same result as with the more laborious method of calculating velocity relative to the center of mass. The ratio of available energies for creating new particles between our two accelerators again turns out the same

$$\eta = \frac{E_1}{E_2} = 452.2 \doteq 452\,.$$

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Problem 44 ... we want to breathe on Everest 7 points

What would be the air pressure at sea level if the Earth had an atmosphere with the same temperature lapse rate as it does today (i.e., a linear decrease of temperature by $0.65\,^\circ\mathrm{C}$ *per* $100\,\mathrm{m}$ *altitude), but at an altitude* $H = 8\,850\,\mathrm{m}$ *above sea level (on Mount Everest), the pressure* p_a *would be the same as it is today at sea level? Consider the sea level temperature* $T_0 = 15\,^\circ\mathrm{C}$.

Karel was thinking about the atmosphere again.

The atmosphere holds together due to the Earth's gravitational force; therefore, we perceive air pressure as hydrostatic pressure. The change in pressure with altitude is

$$\mathrm{d}p = -\rho g\,\mathrm{d}h\,,$$

where ρ is the unknown density of air at the given altitude and the gravitational acceleration g changes very little with altitude, so we consider it to be constant. Furthermore, we know that for air, the state equation holds, from which we express the dependence of density on pressure and temperature as

$$pV = nRT \quad \Rightarrow \quad \rho = \frac{p}{T}\frac{M_\mathrm{m}}{R}\,,$$

where $M_\mathrm{m} = 28.9\,\mathrm{g{\cdot}mol^{-1}}$ is the molar mass of air. We can find it in the tables, on the internet, or express it from the values of normal pressure, density, and temperature in a list of constants. We are also familiar with the dependence of temperature on altitude, which is equal to $T = T_0 - \tau h$, where $\tau = 0.65\,^\circ\mathrm{C}/100\,\mathrm{m}$.

By combining the equations, we get a differential equation for pressure as a function of height

$$\frac{\mathrm{d}p}{p} = -\frac{M_\mathrm{m}g}{RT_0}\cdot\frac{\mathrm{d}h}{1-\frac{\tau}{T_0}h}\,.$$

We integrate the equation and substitute the boundary conditions: the pressure at altitude H is p_a, and the pressure at sea level $0\,\mathrm{m}$ is p_0

$$[\ln p]_{p_0}^{p_\mathrm{a}} = -\frac{M_\mathrm{m}g}{RT_0}\left[\ln\left(1-\frac{\tau}{T_0}h\right)\cdot\left(-\frac{T_0}{\tau}\right)\right]_0^H$$

from where

$$p_0 = p_\mathrm{a}\left(1 - \frac{\tau}{T_0}H\right)^{-\frac{M_\mathrm{m} g}{R\tau}} \doteq 327\,\mathrm{kPa}\,.$$

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Problem 45 ... a lake on a mirror reloaded 7 points

On a table, Jindra laid a hollow spherical mirror with a radius of curvature of $r = 2.00\,\mathrm{m}$ *and a diameter* $D = 12.0\,\mathrm{cm}$. *The optical axis of the mirror points upwards. On the mirror, he poured water with a refractive index* $n = 1.33$ *so that the water formed a surface with a diameter* $d = 6.00\,\mathrm{cm}$. *Jindra illuminated the whole surface of the mirror with rays of light parallel to the optical axis. At what height above the table would an image with the smallest outer diameter possible be created on a hypothetical screen?*

Jindra misses summer swimming.

The focal point of a concave mirror is the point where reflected rays parallel to the optical axis intersect. The focal length is the distance from the focal point to the vertex of the mirror (the point on the mirror's surface lying on the optical axis). In the paraxial approximation, we assume that the angles of all incoming and reflected light rays with the optical axis are small, i.e., $\alpha \ll 1$. Parallel rays to the optical axis hitting the mirror's surface at a perpendicular distance h from the optical axis will reflect at an angle $\alpha \approx 2h/r$. Therefore, in the paraxial approximation, all reflected rays intersect the optical axis at the same distance

$$f_0 = \frac{h}{\alpha} = \frac{r}{2}$$

independent of h. The focal length of a hollow spherical mirror is equal to half the radius of curvature. Therefore, the part of the mirror not covered by water has a focal length of

$$f_0 = \frac{r}{2} = 1.00\,\mathrm{m}.$$

The part of the mirror covered with water has a shorter focal length. The perpendicular water surface does not affect the direction of the rays coming parallel to the optical axis. However, the rays reflected from the mirror strike the water-air interface at an angle to the perpendicular $\alpha \approx \approx 2h/r$, where h is the perpendicular distance from the optical axis. According to Snell's law, these rays break at an angle $\alpha' \approx n\alpha$ and intersect the optical axis closer than the original focal length f_0, at a distance of

$$f = \frac{h}{\alpha'} = \frac{r}{2n} = 0.752\,\mathrm{m}.$$

The light reflected by the outer part of the mirror without water forms a ring of light with an outer diameter of

$$\delta_0 = D\frac{|f_0 - x|}{f_0},$$

where x is the distance of the hypothetical diaphragm from the top of the mirror. The central part of the mirror covered with water then forms a circle of light with a diameter of

$$\delta = d\frac{|f - x|}{f}.$$

The smallest outer diameter of the image occurs at a distance x where $\delta_0 = \delta$. For this position, $f < x < f_0$ necessarily holds. We, therefore, solve the equation

$$D\frac{f_0 - x}{f_0} = d\frac{x - f}{f}$$

for x. Substitute $f = f_0/n$, and we have the equation

$$D\frac{f_0 - x}{f_0} = nd\frac{x - \frac{f_0}{n}}{f_0},$$

which we further manipulate to

$$x(D + nd) = Df_0 + df_0.$$

The solution to the equation is

$$x = \frac{f_0(D + d)}{D + nd},$$

which gives $x = 0.901\,\mathrm{m}$.

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Problem 46 ... coin funnel 7 points

Consider a heavy hemispherical bowl of 19 cm *in diameter and take a point mass that can move in the bowl completely frictionless. We release it from the bowl's edge with velocity* $0.8\,\mathrm{m{\cdot}s^{-1}}$ *in the horizontal direction. What is the minimum height above the bottom of the bowl that the point mass can reach as it travels through the bowl?* *Jarda's wallet spilled out on the street.*

It is always possible to determine the position of a mass point using two parameters. Firstly, the angle θ represents the angle between the position vector of the mass point and the horizontal plane, originating from the center of the upper circle. Secondly, the angle φ describes the translation around the hemisphere's rotational axis from the beginning of the motion. From the initial condition we have $\theta(t = 0) = 0\,^\circ$ and we can put $\varphi(t = 0) = 0\,^\circ$. Moreover, let us denote $R = \frac{19\,\mathrm{cm}}{2} = 9.5\,\mathrm{cm}$.

Two forces act on the mass point during its motion in the hemisphere -– the weight and the bowl's reaction. Looking from above, we can see that the weight always acts downwards, and the bowl's reaction always acts in the direction of the center of the hemisphere. Thus, there is no force that, when viewed from above, acts tangentially on the motion of the point. That implies the law of conservation of the vertical component of the angular momentum of the mass point in relation to the rotational axis of symmetry of the whole hemisphere. We can write it as

$$(\mathbf{L})_z = m\left(\mathbf{r} \times \mathbf{v}\right)_z = mr_x v_y - mr_y v_x\,.$$

Let's consider a rotation of the axes such that $r_y = 0$. Then $r_x = R\cos\theta$. The velocity component v_y is tangent to the horizontal circle on which the mass point currently lies, and the component's magnitude is $v_y = R\cos\theta\dot{\varphi}$, where $\dot{\varphi}$ is the angular velocity about the axis of symmetry of the whole hemisphere. We get the law of conservation of the vertical component of angular momentum as $L_z = mR^2\cos^2\theta\dot{\varphi}$.

Since the mass point is moving without friction, the law of conservation of mechanical energy also holds throughout the problem. The law of conservation of mechanical energy is

$$E = \frac{1}{2}m\left(R^2\dot{\theta}^2 + R^2\cos^2\theta\dot{\varphi}^2\right) - gR\sin\theta\,.$$

Even though we can already calculate the rest of the problem from these two conserved quantities, we still have to determine their values from the initial conditions. For L_z we have $L_z = mR^2\frac{v}{R} = mvR$, while for the energy we have $E = \frac{1}{2}mv^2$. From the conservation law of L_z we substitute $\dot{\varphi}$ into the equation for energy and we get

$$\frac{1}{2}mv^2 = \frac{1}{2}m\left(R^2\dot{\theta}^2 + \frac{v^2}{\cos^2\theta}\right) - gR\sin\theta\,,$$

where we express $\dot{\theta}^2$ as

$$\dot{\theta}^2 = \frac{1}{R^2}\left(2gR\sin\theta - v^2\tan^2\theta\right)\,.$$

Now comes a crucial consideration. For an arbitrarily small initial velocity v, increasing θ will lead to a situation where the right-hand side of the equation is negative due to the properties of the function $\tan x$. However, the left side is the square of a real number, and this side must always be non-negative. The critical angle θ occurs when the angular velocity $\dot{\theta}$ is zero. At this point, its sign changes, and the mass point rises again. Thus, we get the equation

$$2gR\sin\theta_c = v^2\tan^2\theta_c \quad\Rightarrow\quad \sin^2\theta_c + \frac{v^2}{2gR}\sin\theta_c - 1 = 0\,.$$

From this quadratic equation, we find $\sin\theta_c$ as

$$\sin\theta_c = -\frac{v^2}{4gR} + \sqrt{\frac{v^4}{16g^2R^2} + 1}\,,$$

where we chose a positive sign because we expect a positive value of the result. This expression is always less than 1, so we can always find the angle θ_c. This result corresponds to the minimum height

$$h = R\left(1 - \sin\theta_c\right) = R\left(1 + \frac{v^2}{4gR} - \sqrt{\frac{v^4}{16g^2R^2} + 1}\right) = 1.5\,\mathrm{cm}\,,$$

which the mass point can reach.

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Problem 47 ... a point jumps over a cuboid — 7 points

Once upon a time in the land of geometric shapes, a point mass was sitting still on a horizontal plane when it suddenly spotted a cuboid $h = 20.0\,\mathrm{cm}$ tall and $l = 280\,\mathrm{cm}$ long approaching it with a speed $v_k = 6.80\,\mathrm{m\cdot s^{-1}}$. Our poor point mass realizes the time to jump away ran out, so it just jumps over it. What is the minimum speed that the point mass needs to jump in order to be able to jump over the block? The point mass does not only need to jump vertically upwards. *This idea has been in Lego´s ideapad for such a long time*

If the point jumps along the optimal (the one that has the minimum speed) trajectory, it surely touches the upper front edge, stops rising exactly above the center of the cuboid, and touches the upper back edge as it is falling. There is an unlimited amount of possible trajectories satisfying the problem, so we look just for the one that needs the least speed.

When we denote the speed with which the point jumped off the plane v_0 and the speed it will have when touching the front edge of the cuboid v_1, then the law of conservation of energy tells us

$$\frac{1}{2}mv_0^2 = mgh + \frac{1}{2}mv_1^2$$
$$v_0 = \sqrt{2gh + v_1^2}\,,$$

so it is enough to minimize the speed v_1 and plug it all in.

If we start watching the motion of the point mass at the moment it touches the front edge, the conditions that it stops rising exactly above the center of the cuboid and that it touches the back edge falling are equivalent (the reader can prove this mathematically). We will use the first one.

The time it takes for a point to reach the same horizontal position as the center of the cuboid is obtained by dividing their distance (which is at first equal to half the length of the cuboid, or $l/2$) by their relative speed, which is $v_{1x} + v_k$. We denoted the horizontal component of the velocity vector v_{1x}. This is positive if the point jumps toward the block and negative if it jumps backward. Intuitively, to minimize the required speed, one would rather jump so that the mutual speed adds up, but it is not necessary to assume this.

We might obtain the time for the point to stop rising by dividing its vertical velocity component (let's denote it by v_{1y}) by the gravitational acceleration g.

Then the condition that it stops rising above the center of the cuboid can be expressed as the equality of these two times

$$\frac{l/2}{v_{1x} + v_k} = \frac{v_{1y}}{g}\,.$$

This condition therefore limits for which combinations of v_{1x} and v_{1y} the point stops rising exactly above the center of the cuboid. However, we are interested in just one of these combinations which also minimizes the speed $|v_1| = \sqrt{v_{1x}^2 + v_{1y}^2}$. Minimizing the speed is the same as minimizing the speed squared, so we raise the power and get rid of the square root. We plug v_{1y} obtained from the stopping condition and get

$$v_1^2 = v_{1x}^2 + v_{1y}^2 = v_{1x}^2 + \left(\frac{lg}{2}\frac{1}{v_{1x} + v_k}\right)^2\,,$$

where we expressed the quantity to minimize (v_1^2) as a single-variable function of (v_{1x}). Therefore we find the extrema by setting the derivative with respect to the only variable to zero

$$\frac{\mathrm{d}v_1^2}{\mathrm{d}v_{1x}} = 2v_{1x} + \frac{l^2g^2}{4}\frac{-2}{(v_{1x} + v_k)^3} = 0 \qquad \Rightarrow \qquad v_{1x} = \frac{l^2g^2}{4}\frac{1}{(v_{1x} + v_k)^3}\,,$$

which is a quartic equation we really don't want to solve (not even reasoning which one of the 4 solutions truly represents the global minimum), instead, we plot v_1^2 against v_{1x} in a graph 7.

We can clearly see the expression reaches the minimum for $v_1^2 \doteq 3.789\,\mathrm{m^2{\cdot}s^{-2}}$ for $v_{1x} \doteq$ $\doteq 0.487\,\mathrm{m{\cdot}s^{-1}}$ (which matches the intuition that one rather jumps toward). If one were to be

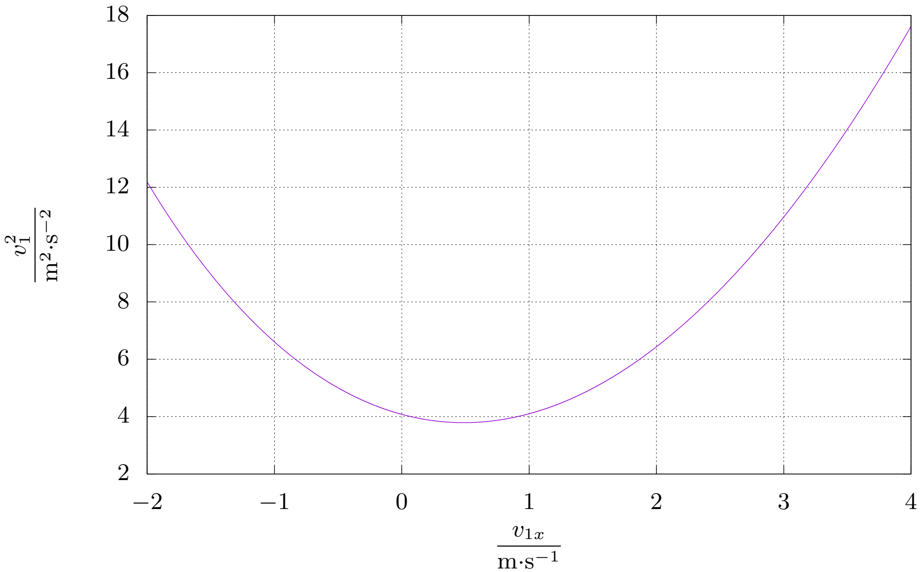

Figure 7: The graph of v_1^2 against v_{1x}

concerned about whether there might not be some even smaller minimum, just remember that $v_1^2 > v_{1x}^2$ must hold, so $|v_{1x}| > \sqrt{3.789\,\mathrm{m^2 \cdot s^{-2}}} \doteq 1.95\,\mathrm{m \cdot s^{-1}}$ is truly the smallest.

Then we just have to remember to plug that back into v_1^2 and into v_0, which gives us that the point must jump off the ground with a speed equal to at least

$$v_0 = \sqrt{2gh + v_1^2} \doteq 2.78\,\mathrm{m \cdot s^{-1}}\,.$$

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Problem 48 ... oscillating lens 8 points

Let's place an isotropic light source on the optical axis of a lens with a focal length of $f = 8.5\,\mathrm{cm}$ at a distance $A = 11\,\mathrm{cm}$ from its center. Then, we attach a spring to the lens, allowing it to perform torsional oscillations. The axis of oscillation is perpendicular to the optical axis of the lens. The moment of inertia of the lens with respect to this axis is $J = 63\,\mathrm{kg \cdot mm^2}$, and the torque acting on the lens is proportional to the angle of rotation from the equilibrium position as $M = -c\varphi$, where $c = 3.7\,\mathrm{mJ}$. What is the maximum speed at which the image of a point of light moves if the lens is deflected $5\,^\circ$ from its equilibrium position and let go?

Jarda wanted to combine optics and oscillation.

Let us first consider what movement the lens makes. From the problem statement, we know the momentum is $M = -c\varphi$, and from the rotational analogy of Newton's second law, we know that the time change of angular momentum is equal to the net torque. We therefore get

$$J\ddot{\varphi} = -c\varphi\,,$$

which is incidentally the equation of a harmonic oscillator. Its solution is

$$\varphi = \varphi_0 \cos(\omega t)\ ,$$

where $\omega = \sqrt{\frac{c}{J}}$ is the angular velocity of the lens' oscillation about its axis.

Using simple geometry, the position of an object relative to time on the lens' optical axis can be determined by the equation $a = A\cos\varphi$. Similarly, the object's distance from the axis can be calculated using the equation $y = A\sin\varphi$. Then, we can use the thin lens formula to display the image of a point located at a distance of $a' = \frac{af}{a-f}$ on the lens axis. Because of the rule that a ray passing through the center of the lens is not refractive, we know that the image will be at the junction of the object and the center of the lens. This line is deflected by an angle φ from the optical axis, and the position of the image is thus at a distance

$$A' = \frac{a'}{\cos\varphi} = \frac{Af}{A\cos\varphi - f}\,.$$

Substituting for the angle φ gives the time dependence of the image's position as

$$A' = \frac{a'}{\cos\varphi} = \frac{Af}{A\cos\left(\varphi_0\cos\left(\sqrt{\frac{c}{J}}t\right)\right) - f}\,.$$

By deriving with respect to time, we find the speed of the image as a function of time

$$V' = \frac{\mathrm{d}A'}{\mathrm{d}t} = \frac{-Af}{\left(A\cos\left(\varphi_0\cos\left(\sqrt{\frac{c}{J}}t\right)\right) - f\right)^2}\left(-A\sin\left(\varphi_0\cos\left(\sqrt{\frac{c}{J}}t\right)\right)\right)\left(-\varphi_0\sqrt{\frac{c}{J}}\sin\left(\sqrt{\frac{c}{J}}t\right)\right)\,.$$

We plot this function in some graphical editor and find that the magnitude of the velocity is maximal at $4.9\,\mathrm{cm\cdot s^{-1}}$.

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Problem 49 ... epic fail 8 points

Radek and Radka are merrily enjoying the ride on the carousel, with Radka sitting a quarter circle in front of Radek (in the direction of rotation). In a moment of mischief, Radka throws a rotten tomato directly at Radek, but after less than half of the carousel's rotation period, the tomato appears back in Radka's face. Determine the magnitude of the velocity (as the ratio of $\kappa > 0$ to the circumferential velocity of the carousel) with which Radka threw the tomato. This grotesque took place in weightlessness. *Radka didn't want to hit... Didn't want to hit herself.*

Since the velocity is to be expressed in units of circumferential velocity, let's set the circumferential velocity equal to 1. Then, κ will be the magnitude of the velocity at which Radka threw the tomato (relative to herself). Since she was throwing directly at Radek from her point of view, the angle that the tomato's velocity vector made with the position vector in the *rotating* system at the moment of the throw must have been equal to $\pi/4$. This position vector marks the boundary of the two half-planes.

If the tomato is to come back to Radka in less than half a period, the velocity of the tomato in the *nonrotating* system at the moment of ejection must be in the opposite half-plane to that of Radek.[2] In this half-plane, the tomato then knocks off Radka.

From now on, we solve the problem in a non-rotating (inertial) frame. At the instant of the ejection, the radial component of the tomato's velocity was equal to $\kappa/\sqrt{2}$, while the tangential component was $1 - \kappa/\sqrt{2}$ (in the direction of rotation). Thus, the angle α that the velocity in the non-rotating system makes at the moment of the tomato's ejection with the position vector satisfies

$$\cos\alpha = \frac{\kappa/\sqrt{2}}{\sqrt{1+\kappa^2-\sqrt{2}\kappa}}\,,$$
$$\sin\alpha = \frac{1-\kappa/\sqrt{2}}{\sqrt{1+\kappa^2-\sqrt{2}\kappa}}\,.$$

Because the magnitude of the velocity of a tomato in a non-rotating system can be calculated from the law of cosine as

$$v = \sqrt{1+\kappa^2-\sqrt{2}\kappa}\,.$$

Let the radius of the carousel also be unitary because the result will certainly not depend on it. The distance the tomato travels before returning to the circumference of the carousel can be found from an isosceles triangle (with vertices Radka - the center of the carousel - the point where the tomato hits Radka, the angle at the first and last vertex being α) as

$$s = 2\cos\alpha\,.$$

So the tomato will return to the perimeter of the carousel in time

$$\tau = \frac{s}{v} = \frac{2\cos\alpha}{\sqrt{1+\kappa^2-\sqrt{2}\kappa}} = \frac{\sqrt{2}\kappa}{1+\kappa^2-\sqrt{2}\kappa}\,.$$

We will now try to manipulate this expression using the formulas for the sine and cosine of a double angle. We find that

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{\sqrt{2}\kappa-\kappa^2}{1+\kappa^2-\sqrt{2}\kappa}$$

and

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{\sqrt{2}\kappa-1}{1+\kappa^2-\sqrt{2}\kappa}\,.$$

[2] This can undoubtedly be achieved by choosing $\kappa < \sqrt{2}$.

Thus, we can rewrite τ as

$$\begin{aligned}\tau = \frac{\sqrt{2}\kappa}{1+\kappa^2-\sqrt{2}\kappa} = \\ = \frac{1+\kappa^2-\sqrt{2}\kappa+\sqrt{2}\kappa-1-\kappa^2+\sqrt{2}\kappa}{1+\kappa^2-\sqrt{2}\kappa} = \\ = 1+\frac{\sqrt{2}\kappa-1-\kappa^2+\sqrt{2}\kappa}{1+\kappa^2-\sqrt{2}\kappa} = \\ = 1+\sin 2\alpha+\cos 2\alpha\,.\end{aligned}$$

If the tomato is to smash against Radka's face, we must also have

$$\tau = \pi - 2\alpha\,.$$

Since the time τ takes Radka to reach the point where her face meets the tomato is due to the unit rotation speed of the carousel corresponding directly to the angle of its rotation. Therefore, after substituting $x = 2\alpha$, we solve the equation

$$1 + x + \sin x + \cos x = \pi\,.$$

The obvious root is $x = \pi$, which gives $\kappa = 0$, i.e., zero speed of the tomato (but this way, the tomato returns to Radka somewhat trivially). The second (and only other) root is found iteratively as

$$x \doteq 0.729\,581\,5\,.$$

From here, after expressing $\tan\alpha$ as a fraction of sin and cosine, we get the speed of a tomato

$$\kappa = \frac{\sqrt{2}}{1+\tan\frac{x}{2}} \doteq 1.023\,397\,.$$

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Problem 50 ... intermittent voltage 7 points

In the circuit, a capacitor with capacitance $C = 47\,\mathrm{nF}$ and a resistor with resistance $R = 220\,\mathrm{k\Omega}$ are connected in series to an intermittent voltage source with period $2T = 20\,\mathrm{ms}$. The voltage waveform over one period is

$$V(t) = \begin{cases} 0\,\mathrm{V} & -T < t < 0; \\ 9.0\,\mathrm{V} & 0 \leq t < T. \end{cases}$$

Determine the difference between the highest and lowest voltage on the capacitor at steady state. *Jindra was playing with a switch in a DC circuit.*

We can describe the relationship between the charge Q and the voltage V_c across the capacitor as

$$Q = CV_\mathrm{c}\,,$$

where C is the capacitance of the capacitor.

At the stage of period $0 \leq t < T$, the voltage $V_0 = 9.0\,\mathrm{V}$ is constant at the source. During this phase, the voltage across the capacitor will increase and it will be the highest when the source switches to the $V = 0\,\mathrm{V}$ stage. At that point the voltage will start to decrease, and it will be the lowest at the moment of switching back to the $V = V_0 = 9.0\,\mathrm{V}$ stage. Let us call the lowest voltage on the capacitor V_- and the highest one V_+.

In the stage when a voltage of the source is $V = V_0$, the charge on the capacitor Q is governed by the differential equation

$$V_0 - \frac{Q}{C} - R\frac{\mathrm{d}Q}{\mathrm{d}t} = 0\,.$$

Using the relationship $Q = CV_\mathrm{c}$ we can rewrite the differential equation for the voltage V_c across the capacitor

$$V_0 - V_\mathrm{c} - RC\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t} = 0\,.$$

The initial condition is $V_\mathrm{c}(0) = V_-$. The solution to this differential equation is a function

$$V_\mathrm{c}(t) = V_0 - (V_0 - V_-)\mathrm{e}^{-\frac{t}{RC}}\,.$$

The equation must hold at time $t = T$, therefore

$$V_+ = V_0 - (V_0 - V_-)\mathrm{e}^{-\frac{T}{RC}}\,.$$

During the zero voltage stage of the source, the charge Q on the capacitor is governed by the differential equation

$$\frac{Q}{C} + R\frac{\mathrm{d}Q}{\mathrm{d}t} = 0\,,$$

which can again be rewritten using the voltage V_c across the capacitor

$$V_\mathrm{c} + RC\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t} = 0\,.$$

The initial condition is $V_\mathrm{c}(0) = V_+$. The solution to this differential equation is a function

$$V_\mathrm{c}(t) = V_+\mathrm{e}^{-\frac{t}{RC}}$$

at time $t = T$, the following must hold

$$V_- = V_+\mathrm{e}^{-\frac{T}{RC}}\,.$$

We get a system of two equations with two unknowns V_-, V_+

$$\begin{aligned} V_- &= V_+\mathrm{e}^{-\frac{T}{RC}}\,, \\ V_+ &= V_0 - (V_0 - V_-)\mathrm{e}^{-\frac{T}{RC}}\,. \end{aligned}$$

From the first equation, we substitute V_- into the second equation and get

$$V_+ = V_0\frac{1}{1 + \mathrm{e}^{-\frac{T}{RC}}}\,.$$

After putting this back into the first equation of V_+, we get

$$V_- = V_0 \frac{\mathrm{e}^{-\frac{T}{RC}}}{1+\mathrm{e}^{-\frac{T}{RC}}}\,.$$

The difference between the highest and lowest voltage on the capacitor is

$$V_+ - V_- = V_0 \frac{1-\mathrm{e}^{-\frac{T}{RC}}}{1+\mathrm{e}^{-\frac{T}{RC}}}.$$

After substituting the numbers from the problem statement, we get the difference $4.04\,\mathrm{V} \doteq \doteq 4.0\,\mathrm{V}$.

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Problem 51 ... Van Allenovi is just bright 8 points

In the equatorial plane, at a distance of two Earth radii from the center of Earth, there is a proton with energy $E = 1\,\mathrm{keV}$, which is pointing at an angle $\alpha = 45^\circ$ from the force line towards the North Pole. As it approaches Earth, at one moment it is reflected by a magnetic mirror and returns towards the South Pole. Determine the (magnetic) latitude at which this happens. Treat the Earth's magnetic field as a dipole.

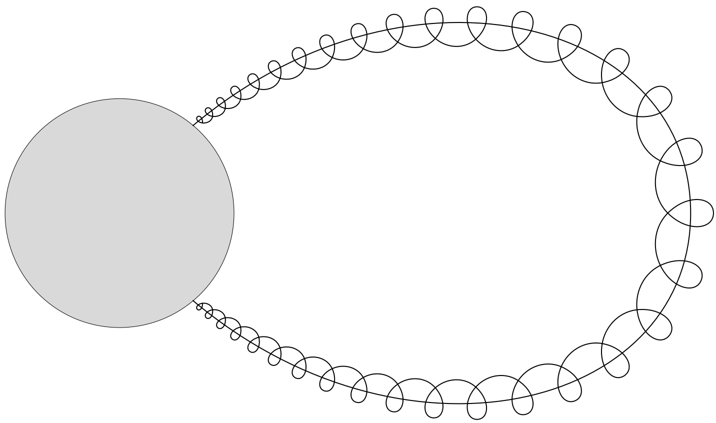

Figure 8: Representation of proton behavior.

Kačka got the result, but wanted to check it.

A proton in the Earth's magnetic field performs several motions, the simplest is the cyclotron motion around the magnetic field line, that is, the part of the motion perpendicular to the field line. In the direction parallel to the field line, the particle moves uniformly. However, as the proton approaches the pole along the magnetic field line, the amplitude of the magnetic induction increases because, in this case, it must maintain its magnetic moment. Due to the conservation of the magnetic moment $\mu = mv_\perp^2/(2B)$ the velocity parallel to the magnetic field changes to a velocity perpendicular to the magnetic field. Earth's magnetic field is considered a dipole, the expression of the magnetic field in polar coordinates is $\mathbf{B}(r,\theta,\varphi) = \left(B_0 R_E^3/r^3\right)(2\cos\theta, \sin\theta, 0)$ and the shape of the magnetic field line, which is in the equatorial plane in a distance L from the center of Earth, is $r(\theta) = L\sin^2(\theta)$. This gives us the condition for reflection, from the conservation of the magnetic moment we write:

$$\mu = \frac{mv_\perp^2}{2B(2R_Z, 0)} = \frac{mv^2}{2B(2R_z\sin^2\theta_r, \theta_r)}.$$

We substitute for the magnetic field and express the angle θ_r:

$$\begin{aligned} \frac{mv^2\sin^2\alpha}{2B_0\frac{R_E^3}{(2R_E)^3}} &= \frac{mv^2}{2B_0\frac{R_E^3}{(2R_E\sin^2\theta_r)^3}\sqrt{4\cos^2\theta_r + \sin^2\theta_r}}\,, \\ \sin^2\alpha &= \frac{\sin^6\theta_r}{\sqrt{4\cos^2\theta_r + \sin^2\theta_r}}\,, \\ \sin^2\alpha\sqrt{4 - 3\sin^2\theta_r} &= \sin^6\theta_r\,, \\ \sin^4\alpha\left(4 - 3\sin^2\theta_r\right) &= \sin^{12}\theta_r\,. \end{aligned}$$

Using the substitution $x = \sin^2\theta_r$ we get the equation $\sin^4\alpha\,(4 - 3x) = x^6$, which is analytically unsolvable. However, when we substitute the known angle $\alpha = 45\,^\circ$ then $\sin^2\alpha = 0.5$. The problem can be solved numerically, yielding a positive result of $\sin^2\theta_r = 0.846 \to \theta_r = 66.87°$. The magnetic width is then the complementary angle, $23.13\,^\circ$.

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Problem 52 ... flying mud — 8 points

A tire with an outer diameter $d = 63.2\,\mathrm{cm}$ *is rolling on a flat surface at a constant speed* $v = 15.0\,\mathrm{m{\cdot}s^{-1}}$. *Suddenly, a piece of mud gets ejected from its rotating circumference. The soil flies through the air and eventually lands on the ground. Subsequently, the tire passes over the fallen piece of mud with the same part from which it was ejected. How long was the soil on the ground before the tire ran over it? Disregard air resistance during the flight of the mud. The mud did not fall farther than* $4.00\,\mathrm{m}$ *along the horizontal axis from the release point.*

While driving, a fallen tractor wheel whizzed past Jindra.

We will measure the angle α on the tire from the vertical direction. In the reference frame connected to the ground, the mud has a velocity vector

$$\mathbf{u} = v\,(1 + \cos\alpha, -\sin\alpha)\,.$$

When the piece of mud separates from the wheel, the mud with this initial velocity vector will continue in free fall toward the ground. However, we currently do not know the angle α that determines the location of the mud separation. Therefore, we will derive an equation to calculate the angle α. The piece of mud begins its fall to the ground from a height

$$h = r(1 + \cos\alpha)\,,$$

where $r = 31.6\,\mathrm{cm}$ is the radius of the tire. Its vertical position y and horizontal position x relative to the release point are

$$\begin{aligned} y &= -vt\sin\alpha - \frac{1}{2}gt^2\,,\\ x &= v\ (1+\cos\alpha)t\,. \end{aligned}$$

We will express the time $t = x/(v(1+\cos\alpha))$ from the second equation and substitute it into the first equation, resulting in a dependence $y = y(x)$

$$y = -\frac{\sin\alpha}{1+\cos\alpha}x - \frac{g}{2v^2(1+\cos\alpha)^2}x^2.$$

The mud hits the ground when $y = -h = -r(1+\cos\alpha)$, thus obtaining a relationship between the impact location x and the separation angle α

$$\begin{aligned} \frac{g}{2v^2(1+\cos\alpha)^2}x^2 + \frac{\sin\alpha}{1+\cos\alpha}x - r(1+\cos\alpha) &= 0\,,\\ \frac{1}{2}x^2 + \frac{v^2\sin\alpha(1+\cos\alpha)}{g}x - \frac{rv^2(1+\cos\alpha)^3}{g} &= 0\,. \end{aligned}$$

The roots of this equation are

$$x_{1,2} = -\frac{v^2\sin\alpha(1+\cos\alpha)}{g} \pm \sqrt{\left(\frac{v^2\sin\alpha(1+\cos\alpha)}{g}\right)^2 + \frac{2rv^2(1+\cos\alpha)^3}{g}}\,.$$

We are interested only in the positive root

$$x_1 = (1+\cos\alpha)\left(-\frac{v^2\sin\alpha}{g} + \sqrt{\left(\frac{v^2\sin\alpha}{g}\right)^2 + \frac{2v^2r(1+\cos\alpha)}{g}}\right)\,.$$

It is important to note that the current x-position is relative to the point of mud separation. However, we need to find the x-position relative to the point where the wheel contacts the ground at the time of mud separation. To achieve this, we must calculate the separation point's x-coordinate in relation to the point of contact, which can be expressed as $r\sin\alpha$.

For the wheel to run over the piece of mud at the same spot from which it previously separated, it must rotate by an angle $\pi - \alpha$ and complete k full revolutions. The mud must fall at a distance $r(2k\ \pi + \pi - \alpha)$ from the wheel's point of contact with the ground. Therefore, we obtain an equation for the separation angle

$$r(2k\ \pi + \pi - \alpha) = r\sin\alpha + (1+\cos\alpha)\left(-\frac{v^2\sin\alpha}{g} + \sqrt{\left(\frac{v^2\sin\alpha}{g}\right)^2 + \frac{2v^2r(1+\cos\alpha)}{g}}\right)\,,$$

which we have to solve numerically. If we divide both sides of the equation by r, we can introduce a dimensionless parameter $A = v^2/(gr)$, which will simplify the equation

$$(2k+1)\pi = \alpha + \sin\alpha + (1+\cos\alpha)\left(-A\sin\alpha + \sqrt{(A\ \sin\alpha)^2 + 2A\ (1+\cos\alpha)}\right)\,.$$

After substituting $A = 72.58$ according to the given numbers for various k, we find (for example, using the function `scipy.optimize.fsolve()` in Python) numerical solution

$$\begin{aligned} k &= 0, \quad & \alpha &= \pi\,\mathrm{rad}, & x_1 &= 0\,\mathrm{m},\\ k &= 1, \quad & \alpha &= 0.4070\,\mathrm{rad}, & x_1 &= 2.72\,\mathrm{m},\\ k &= 2, \quad & \alpha &= 0.2042\,\mathrm{rad}, & x_1 &= 4.84\,\mathrm{m}.\\ &\vdots & &\vdots & &\vdots \end{aligned}$$

The case $k \geq 2$ does not satisfy the condition from the task that the mud landed closer than 4.00 m horizontally from the release point. The case $k = 0$ is again unsatisfactory because the soil did not fly through the air. It is a trivial case where the mud remained on the tire. The only possible solution is for $k = 1$.

The wheel traveled a distance $D = r(2\pi + \pi - \alpha)$ in time

$$t_k = \frac{D}{v} = 0.1900\,\mathrm{s}\,.$$

The mud was flying through the air for a duration before hitting the ground

$$t = \frac{x_1}{v(1+\cos\alpha)} = 0.0947\,\mathrm{s}\,.$$

The time period when the mud lay on the ground before being run over again was $T = t_k - t = 0.0953\,\mathrm{s}$

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Problem 53 ... maximal activity II — 9 points

Jindra has $N_0 = 10^9$ atoms of the radioactive isotope ^{212}Bi. It decays to the isotope ^{212}Po with probability $P_\beta = 64.06\%$ by beta decay and to the isotope ^{208}Tl with probability $P_\alpha = 35.94\%$ by alpha decay. The half-life of bismuth is $T_{\mathrm{Bi}} = 60.6\,\mathrm{min}$. The polonium isotope decays further by alpha decay with a half-life of $T_{\mathrm{Po}} = 299\,\mathrm{ns}$ to the stable isotope ^{208}Pb. The thallium isotope decays by beta decay with a half-life of $T_{\mathrm{Tl}} = 3.05\,\mathrm{min}$ also to lead ^{208}Pb.

How long does it take Jindra to measure the maximum activity in the system?

Jindra measured the half-life of ^{212}Po.

For the purpose of solving the problem, it will be easier to work with decay constants

$$\lambda = \frac{\ln 2}{T_{1/2}}$$

than with half-lives $T_{1/2}$. The decay constants for each isotope are

$$\lambda_{\mathrm{Bi}} = 1.91\cdot 10^{-4}\,\mathrm{s^{-1}}, \quad \lambda_{\mathrm{Po}} = 2.32\cdot 10^{6}\,\mathrm{s^{-1}}, \quad \lambda_{\mathrm{Tl}} = 3.79\cdot 10^{-3}\,\mathrm{s^{-1}}.$$

For clarity, we use the variable A for the number of bismuth atoms, B for the polonium atoms, C for the thallium atoms, and D for the lead atoms. The total activity R in the system depends on the actual amounts of the isotopes of bismuth A, polonium B, and thallium C.

$$R = \lambda_{\mathrm{Bi}}A + \lambda_{\mathrm{Po}}B + \lambda_{\mathrm{Tl}}C. \tag{8}$$

Therefore, we must solve a system of differential equations describing the decay series to find how A, B, and C evolve in time. The system of differential equations describing the amount of each isotope over time is

$$\begin{aligned}\dot{A} &= -\lambda_{\mathrm{Bi}} A,\\ \dot{B} &= P_\beta \lambda_{\mathrm{Bi}} A \ - \lambda_{\mathrm{Po}} B,\\ \dot{C} &= P_\alpha \lambda_{\mathrm{Bi}} A \ - \lambda_{\mathrm{Tl}} C,\\ \dot{D} &= \lambda_{\mathrm{Po}} B + \lambda_{\mathrm{Tl}} C,\end{aligned}$$

with initial conditions $A(0) = N_0, B(0) = 0, C(0) = 0, D(0) = 0$.

This system of differential equations can be solved from the top line by line. The first line of the equation has the solution

$$A(t) = N_0 \mathrm{e}^{-\lambda_{\mathrm{Bi}} t}.$$

Add this to the second line and solve the differential equation

$$\dot{B} = P_\beta \lambda_{\mathrm{Bi}} N_0 \mathrm{e}^{-\lambda_{\mathrm{Bi}} t} - \lambda_{\mathrm{Po}} B$$

with initial condition $B(0) = 0$. The solution is the function

$$B(t) = \frac{P_\beta \lambda_{\mathrm{Bi}} N_0}{\lambda_{\mathrm{Po}} - \lambda_{\mathrm{Bi}}} \left(\mathrm{e}^{-\lambda_{\mathrm{Bi}} t} - \mathrm{e}^{-\lambda_{\mathrm{Po}} t}\right)$$

The differential equation on the third line has the same structure as the equation on the second line and the same initial condition, so its solution is

$$C(t) = \frac{P_\alpha \lambda_{\mathrm{Bi}} N_0}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}} \left(\mathrm{e}^{-\lambda_{\mathrm{Bi}} t} - \mathrm{e}^{-\lambda_{\mathrm{Tl}} t}\right)$$

We plug the derived functions into the equation (8) for the activity

$$\begin{aligned}R(t) = \lambda_{\mathrm{Bi}} N_0 \mathrm{e}^{-\lambda_{\mathrm{Bi}} t} + \frac{P_\beta \lambda_{\mathrm{Bi}} \lambda_{\mathrm{Po}} N_0}{\lambda_{\mathrm{Po}} - \lambda_{\mathrm{Bi}}} \left(\mathrm{e}^{-\lambda_{\mathrm{Bi}} t} - \mathrm{e}^{-\lambda_{\mathrm{Po}} t}\right) + \\ + \frac{P_\alpha \lambda_{\mathrm{Bi}} \lambda_{\mathrm{Tl}} N_0}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}} \left(\mathrm{e}^{-\lambda_{\mathrm{Bi}} t} - \mathrm{e}^{-\lambda_{\mathrm{Tl}} t}\right)\end{aligned} \tag{9}$$

We verify that in the limit $t \to \infty$, the activity goes to zero $R \to 0$ due to decreasing exponentials. This is consistent with our expectation, since in a long time, most of the radioactive atoms decay and only stable ^{208}Pb remain.

To find the maximum of the activity, we have to derive the function (9)

$$\begin{aligned}\dot{R} = \lambda_{\mathrm{Bi}} N_0 \Bigg(- \lambda_{\mathrm{Bi}} \mathrm{e}^{-\lambda_{\mathrm{Bi}} t} - \frac{P_\beta \lambda_{\mathrm{Bi}} \lambda_{\mathrm{Po}}}{\lambda_{\mathrm{Po}} - \lambda_{\mathrm{Bi}}} \mathrm{e}^{-\lambda_{\mathrm{Bi}} t} + \frac{P_\beta \lambda_{\mathrm{Po}}^2}{\lambda_{\mathrm{Po}} - \lambda_{\mathrm{Bi}}} \mathrm{e}^{-\lambda_{\mathrm{Po}} t} - \\ - \frac{P_\alpha \lambda_{\mathrm{Bi}} \lambda_{\mathrm{Tl}}}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}} \mathrm{e}^{-\lambda_{\mathrm{Bi}} t} + \frac{P_\alpha \lambda_{\mathrm{Tl}}^2}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}} \mathrm{e}^{-\lambda_{\mathrm{Tl}} t} \Bigg).\end{aligned} \tag{10}$$

Let's see that at time $t = 0$ the change in activity over time is positive

$$\dot{R}(0) = \lambda_{\mathrm{Bi}} N_0 \left(-\lambda_{\mathrm{Bi}} + P_\beta \lambda_{\mathrm{Po}} + P_\alpha \lambda_{\mathrm{Tl}}\right) > 0,$$

therefore, initially, the decay activity in the system increases. At some point, it reaches a maximum, and then in the limit $t \to \infty$, the activity drops to zero.

We determine the time of maximum activity by setting the derivative of the activity with respect to time (10) equal to zero and solving for time t. Due to the fact $\lambda_{Po} \gg \lambda_{Tl}$ and $\lambda_{Po} \gg \lambda_{Bi}$, we can neglect the term with the exponential $\mathrm{e}^{-\lambda_{Po} t}$ in the equation.

$$-\lambda_{\mathrm{Bi}}\mathrm{e}^{-\lambda_{\mathrm{Bi}}t} - \frac{P_\beta \lambda_{\mathrm{Bi}}\lambda_{\mathrm{Po}}}{\lambda_{\mathrm{Po}} - \lambda_{\mathrm{Bi}}}\mathrm{e}^{-\lambda_{\mathrm{Bi}}t} - \frac{P_\alpha \lambda_{\mathrm{Bi}}\lambda_{\mathrm{Tl}}}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}\mathrm{e}^{-\lambda_{\mathrm{Bi}}t} + \frac{P_\alpha \lambda_{\mathrm{Tl}}^2}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}\mathrm{e}^{-\lambda_{\mathrm{Tl}}t} = 0$$

We gradually isolate the time t

$$\mathrm{e}^{(\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}})t} = \frac{\frac{P_\alpha \lambda_{\mathrm{Tl}}^2}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}}{\lambda_{\mathrm{Bi}} + \frac{P_\beta \lambda_{\mathrm{Bi}}\lambda_{\mathrm{Po}}}{\lambda_{\mathrm{Po}} - \lambda_{\mathrm{Bi}}} + \frac{P_\alpha \lambda_{\mathrm{Bi}}\lambda_{\mathrm{Tl}}}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}} \approx \frac{\frac{P_\alpha \lambda_{\mathrm{Tl}}^2}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}}{\lambda_{\mathrm{Bi}} + P_\beta \lambda_{\mathrm{Bi}} + \frac{P_\alpha \lambda_{\mathrm{Bi}}\lambda_{\mathrm{Tl}}}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}}$$

$$t = \frac{1}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}} \ln\left(\frac{\frac{P_\alpha \lambda_{\mathrm{Tl}}^2}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}}{\lambda_{\mathrm{Bi}} + P_\beta \lambda_{\mathrm{Bi}} + \frac{P_\alpha \lambda_{\mathrm{Bi}}\lambda_{\mathrm{Tl}}}{\lambda_{\mathrm{Tl}} - \lambda_{\mathrm{Bi}}}} \right).$$

After plugging in the numbers, the time of maximum activity in the system comes out $t = 366\,\mathrm{s} = 6.09\,\mathrm{min}$.

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Problem 54 ... Mišo is shooting 9 points

Mišo likes to shoot from a laser. He needs to reduce its energy with reflective gray (neutral density, ND) filters. He would like to achieve a beam with an energy 37 J. *Mišo has 5 filter holders and 7 different filters, namely 2-stop ND, 3-stop ND, 5-stop ND, 7-stop ND, 11-stop ND, 13-stop ND, and 17-stop ND. Assume that all energy striking the filter is either reflected or transmitted. The laser has an energy of* 77 377 J. *With what accuracy can he achieve the desired* 37 J*? Give the result in* mJ. *Mišo was calculating the filtering at the PALS*

There will be an infinite number of reflections in the spaces between the filters. One possibility would be to compute infinite series. However, this is not necessary. We will be interested in the total amount of energy flowing in the spaces in between the filters. We denote the input energy 77 377 J by E and we assume that the laser is shining from the left. Firstly, let's suppose that we have used all five holders. We denote all the filters starting from the left by the indices 1 to 5. We then denote by the same indices the energies E flowing from the given filters towards the right and the returning energies R flowing into the given filters from the right. The resulting energy coming out of the system of filters will be E_5.

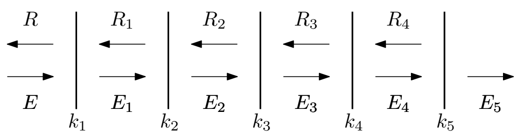

Figure 9: Energy in the spaces between the filters

The number n in the ND filter label indicates how much light passes through the filter

$$E_{\text{passes}} = E_{\text{enters}} \cdot 2^{-n} \,.$$

We substitute $k = 2^{-n}$ and then for the coefficient k we can get

$$k \;= \left\{\frac{1}{4}, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}, \frac{1}{2\,048}, \frac{1}{8\,192}, \frac{1}{131\,072}\right\}, .$$

Energy is not lost in the filters, so

$$E_{\text{reflect}} = E_{\text{enters}} \cdot (1 - k_i) \quad , \quad E_{\text{reflect}} + E_{\text{passes}} = E_{\text{enters}} \,.$$

We place five filters with coefficients k_1, k_2, k_3, k_4, k_5 whose values can be any combination from the set k in the holders. The following equations will hold on the interfaces

$$\begin{aligned}
E_5 &= k_5 E_4 \,,\\
E_4 &= k_4 E_3 + (1 - k_4) R_4 \,,\\
E_3 &= k_3 E_2 + (1 - k_3) R_3 \,,\\
E_2 &= k_2 E_1 + (1 - k_2) R_2 \,,\\
E_1 &= k_1 E + (1 - k_1) R_1 \,,\\
R_4 &= (1 - k_5) E_4 \,,\\
R_3 &= (1 - k_4) E_3 + k_4 R_4 \,,\\
R_2 &= (1 - k_3) E_2 + k_3 R_3 \,,\\
R_1 &= (1 - k_2) E_1 + k_2 R_2 \,,
\end{aligned}$$

where E_i are the energies flowing to the right and R_i are the energies flowing to the left.

We got nine equations with nine unknowns (5 times E_i and 4 times R_i), which we will solve using the matrix notation and Cramer's rule. The matrix notation of the equations above is as follows

$$\underbrace{\begin{pmatrix}
1 & -k_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & -k_4 & 0 & 0 & k_4 - 1 & 0 & 0 & 0\\
0 & 0 & 1 & -k_3 & 0 & 0 & k_3 - 1 & 0 & 0\\
0 & 0 & 0 & 1 & -k_2 & 0 & 0 & k_2 - 1 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & k_1 - 1\\
0 & k_5 - 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & k_4 - 1 & 0 & 0 & -k_4 & 1 & 0 & 0\\
0 & 0 & 0 & k_3 - 1 & 0 & 0 & -k_3 & 1 & 0\\
0 & 0 & 0 & 0 & k_2 - 1 & 0 & 0 & -k_2 & 1
\end{pmatrix}}_{\text{Túto maticu označme } A}
\begin{pmatrix} E_5\\ E_4\\ E_3\\ E_2\\ E_1\\ R_4\\ R_3\\ R_2\\ R_1 \end{pmatrix}
=
\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ k_1 E\\ 0\\ 0\\ 0\\ 0 \end{pmatrix} .$$

For Cramer's rule, we need the determinant of the matrix A and the determinant of the matrix A_1 where we replace the first column with the vector on the right-hand side of the equation. We replace the first one because we only want to find E_5, which is the first unknown in our ordering. We get

$$E_5 = \frac{\det A_1}{\det A} = E \cdot \frac{k_1 k_2 k_3 k_4 k_5}{k_1 k_2 k_3 k_4 + k_1 k_2 k_3 k_5 + k_1 k_2 k_4 k_5 + k_1 k_3 k_4 k_5 + k_2 k_3 k_4 k_5 - 4 k_1 k_2 k_3 k_4 k_5} \,,$$

which can be simplified as

$$E_5 = E \cdot \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4} + \frac{1}{k_5} - 4} = \frac{1}{2^{n_1} + 2^{n_2} + 2^{n_3} + 2^{n_4} + 2^{n_5} - 4},$$

Where n_i are the stop numbers of the ND filters.

By an analogous process for $N = \{1, 2, 3, 4, 5\}$ filters, we obtain in general

$$E_N = E \cdot \left(1 - N + \sum_{i=1}^{N} 2^{n_i}\right)^{-1}, \tag{11}$$

where E_N is the resulting energy leaving the system of N filters. This relation likely holds for any N. We can see that the resulting energy is not dependent on the order of the filters in the holders. All permutations for a given combination are the same. Our goal now will be to find a combination whose energy is as close to 37 J as possible, according to the relation (11).

From the relation (11), we can express and calculate the sum of the terms 2^{n_i}, putting $E_N = 37\,\mathrm{J}$, which is the desired output energy to which we want to get as close as possible.

$$\sum_{i=1}^{N} 2^{n_i} = \frac{E}{E_N} + N - 1 \doteq 2090 + N\,. \tag{12}$$

Now we can go through all the combinations using a script and find the best one. However, we can also obtain the result by simple reasoning.

We have the terms $2^{n_i} = 1/k_i$, so 4, 8, 32, 128, 2 048, 8 192, 131 072, to "make up" the number $2\,090 + N$. So we need to use the number $2\,048 = 2^{11}$, which is an 11-stop filter. Higher ND filters are too strong. That leaves $42 + N$ left from the sum, so 128 is too much, and we use a 5-stop filter, which corresponds to $2^5 = 32$. That leaves $10 + N$. The best we can do is to use the remaining 2 filters that we haven't eliminated yet. That is the 2-stop ND and the 3-stop ND, which together give $2^2 + 2^3 = 4 + 8 = 12$. We have used 4 filters, which means that $N = 4$. This gives us 2094 on the right-hand side of the equation (12), which is almost exactly equal to the sum on the left-hand side, which we determined to be $2\,048 + 32 + 8 + 4 = 2\,092$.

Finally, we calculate the deviation of ΔE energy E_N from 37 J using the relation (11) for the best set of filters, namely 2-stop ND, 3-stop ND, 5-stop ND and 11-stop ND,

$$\Delta E = 77\,377\,\mathrm{J} \cdot \left(1 - 4 + 2^2 + 2^3 + 2^5 + 2^{11}\right)^{-1} - 37\,\mathrm{J} \doteq 40\,\mathrm{mJ}$$

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Problem 55 ... Red Wednesday — 9 points

Lego is situated aboard a space station in deep space. To avoid going crazy, he keeps watching what is the day today on Earth. He knows that last Sunday he launched a rocket (at rest) with a rest mass $m_0 = 31.0\,\mathrm{kg}$ propulsed by an ion thruster which does not reduce the rest mass; however, it exerts a constant force F. The rocket glows with a golden light $\lambda_0 = 600\,\mathrm{nm}$ in such a way that when Lego points his telescope at it today at midday (specifically $t_\mathrm{s} = 72.0\,\mathrm{h}$ since the launch), he sees a red light $\lambda_r = 670\,\mathrm{nm}$. How big is the force F?

Lego spotted a pattern in the date of Physics Brawl Online...

Redshift is a well-known case of a Doppler effect. To be specific, there is a relation between the emitted light's wavelength λ_0 and the received wavelength λ_r

$$\lambda_r = \lambda_0 \frac{\sqrt{1+v/c}}{\sqrt{1-v/c}}\,,$$

where v is the speed at which the source and the observer move mutually away and c is the speed of light. The fraction v/c is commonly denoted as β.

When we take the square and isolate β, we obtain

$$\beta = \frac{(\lambda_r/\lambda_0)^2 - 1}{(\lambda_r/\lambda_0)^2 + 1}\,,$$

which gives us the speed at which the rocket is receding from Lego's space station $v = 0.1c$.

Let's take a look at how the rocket will accelerate. Considering the acceleration up to $v = 0.1c$, we have to take relativistic equations into account for acceleration. To be precise, the spatial component of the motion equation looks the same as the classical one $\vec{F} = \mathrm{d}\vec{p}/\,\mathrm{d}t$, except $\vec{p} = m\vec{v}$, where $\vec{v}$ is a classical velocity relative to a certain inertial frame of reference. However, the mass m is relative to this reference frame by $m = m_0\gamma$, where m_0 is the (already given in the problem) rest mass and $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor.

We choose the frame of reference associated with the station at which Lego is aboard to solve this problem. All distances, times, and speeds will be given in terms of this frame.

Now we can construct a differential equation for v and solve, but it is easier to realize that the momentum at time t will simply be $p(t) = Ft$, which gives us the relation between t and β

$$\begin{aligned} Ft &= m_0 \frac{1}{\sqrt{1-\beta^2}} v \\ \frac{Ft}{m_0 c} &= \sqrt{\frac{\beta^2}{1-\beta^2}} \\ \left(\frac{Ft}{m_0 c}\right)^2 (1-\beta^2) &= \beta^2 \\ \beta &= \frac{Ft}{\sqrt{(m_0 c)^2 + (Ft)^2}}\,. \end{aligned}$$

When we substitute β calculated from the redshift, we get

$$\frac{Ft}{\sqrt{(m_0 c)^2 + (Ft)^2}} = \frac{(\lambda_r/\lambda_0)^2 - 1}{(\lambda_r/\lambda_0)^2 + 1}\,.$$

It is crucial to realize that it is not enough to plug in the time t_s here since the light that Lego sees on Wednesday departed the rocket earlier. Light moves at the speed c, so if the rocket is at some point (from the perspective of the frame associated with Lego's station) x away from Lego, it will reach Lego in the time $t_\mathrm{c} = x/c$. That means Lego launched the rocket at the time $t = 0$. After some time $t = t_\mathrm{r}$, it will gain such a speed that it will be seen as red by Lego. But Lego will see it with a time delay t_c. So for Lego to be able to see this given wavelength at the time t_s, it must hold that $t_\mathrm{s} = t_\mathrm{r} + t_\mathrm{c}$.

Firstly, we will express t_r from the equation obtained by comparing β

$$\frac{1}{\sqrt{\left(\frac{m_0 c}{F t_\mathrm{r}}\right)^2+1}}=\frac{(\lambda_r/\lambda_0)^2-1}{(\lambda_r/\lambda_0)^2+1}$$

$$\frac{1}{\left(\frac{m_0 c}{F t_\mathrm{r}}\right)^2+1}=\left(\frac{(\lambda_r/\lambda_0)^2-1}{(\lambda_r/\lambda_0)^2+1}\right)^2$$

$$\left(\frac{(\lambda_r/\lambda_0)^2+1}{(\lambda_r/\lambda_0)^2-1}\right)^2-1=\left(\frac{m_0 c}{F t_\mathrm{r}}\right)^2$$

$$t_\mathrm{r}=\frac{m_0 c}{F}\frac{(\lambda_r/\lambda_0)^2-1}{2\lambda_r/\lambda_0}\,.$$

Secondly, we calculate the distance traveled by the rocket at that time. We know its speed at the time t, so we just need to integrate from 0 to t_r

$$x_r=\int_0^{t_\mathrm{r}}\frac{c}{\sqrt{\left(\frac{m_0 c}{F t}\right)^2+1}}\,\mathrm{d}t=c\left[\sqrt{t^2+\left(\frac{m_0 c}{F}\right)^2}\right]_0^{t_\mathrm{r}}=c\left(\sqrt{t_\mathrm{r}^2+\left(\frac{m_0 c}{F}\right)^2}-\frac{m_0 c}{F}\right)\,.$$

Thus, the time for the light to return will be

$$t_\mathrm{c}=\frac{x_r}{c}=\sqrt{t_\mathrm{r}^2+\left(\frac{m_0 c}{F}\right)^2}-\frac{m_0 c}{F}=\frac{m_0 c}{F}\frac{(\lambda_r/\lambda_0)^2+1}{2\lambda_r/\lambda_0}-\frac{m_0 c}{F}=\frac{m_0 c}{F}\frac{(\lambda_r/\lambda_0-1)^2}{2\lambda_r/\lambda_0}\,.$$

And as we have said, $t_\mathrm{s}=t_\mathrm{r}+t_\mathrm{c}$ must hold, which finally gives us the equation for F

$$\begin{aligned}
t_\mathrm{s}&=\frac{m_0 c}{F}\frac{(\lambda_r/\lambda_0)^2-1}{2\lambda_r/\lambda_0}+\frac{m_0 c}{F}\frac{(\lambda_r/\lambda_0-1)^2}{2\lambda_r/\lambda_0}\,,\\
F&=\frac{m_0 c}{t_\mathrm{s}}\left(\frac{(\lambda_r/\lambda_0)^2-1}{2\lambda_r/\lambda_0}+\frac{(\lambda_r/\lambda_0)^2-2\lambda_r/\lambda_0+1}{2\lambda_r/\lambda_0}\right)\,,\\
F&=\frac{m_0 c}{t_\mathrm{s}}\left(\lambda_r/\lambda_0-1\right)=4\,183\,\mathrm{N}\,.
\end{aligned}$$

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Problem 56 ... pumping water using water 8 points

In the 19th century, people began to make extensive use of steam engines and to think about their efficiency. One of the problems they could solve back then was pumping water out of the mine. However, we are not interested in the specific design of a process/engine that pumps water from a depth of $h=50\,\mathrm{m}$ to the surface. Consider that we have stumbled upon geothermally heated ideal water (it is incompressible, has a constant density ρ and specific heat capacity c) with volume $V_1=200\,\mathrm{m^3}$ and temperature $T_1=90\,\mathrm{°C}$. Nearby is also a lake (thermal bath) with constant temperature $T_\mathrm{J}=10\,\mathrm{°C}$. What is the maximum amount of water we can ideally pump to the surface? For the whole amount of water, consider a constant elevation h and a homogeneous gravitational field. *Marek made an excursion to the 19th century.*

From the problem statement, we can see that it is a problem where we have a certain amount of energy (geothermal water) available, which we want to convert into a certain amount of work (pumping water from a mine). Whatever we do, we consider that the total energy is conserved.

We are interested in the maximum amount of water we can pump out and, therefore, the maximum amount of work we can do. Let's do things efficiently and consider the ideal case where the geothermal energy is converted only into work and "waste" energy, which is received by the lake (a thermal bath of constant temperature T_J). At this point, let us note that the lake does have to accept a certain amount of heat/energy in this process, and thus, we cannot convert geothermal energy purely into work (this would violate the second law of thermodynamics). So, let's write the law of conservation of energy

$$\Delta Q_\mathrm{geo} + \Delta Q_\mathrm{J} + W = 0\,, \tag{13}$$

where ΔQ_geo denotes the heat/energy change of the geothermal water, ΔQ_J denotes the heat the lake receives, and W is the work we are looking for. We see that the problem actually reduces to finding and minimizing just the heat ΔQ_J, since ΔQ_geo is given by

$$\Delta Q_\mathrm{geo} = cV_1\rho\Delta T = cV_1\rho(T_\mathrm{J} - T_1)\,,$$

where the temperature at the beginning is set, and the temperature at the end is the temperature of the lake itself. This is because we want maximum work, so we will cool the geothermal water as long as we can efficiently (until the temperatures equilibrate, it would cost us work).

In the relation (13), we are left with two unknowns, so we need one more relation. And that is the condition that the whole process must be reversible! We know from Carnot that it is processes operating between two temperatures that are the most efficient – they can extract the most work. The law of energy conservation and the condition of reversibility of the whole process are at the heart of the so-called "Maximum work" theorem. The second law of thermodynamics in the form

$$\Delta S \ \geq 0\,,$$

respectively, for our process,

$$\Delta S_\mathrm{geo} + \Delta S_\mathrm{J} + \Delta S_W \geq 0\,. \tag{14}$$

And since a bath, by definition (being much larger than a geothermally heated water source) receives heat at a constant temperature, the entropy change is

$$\Delta S_\mathrm{J} = \frac{\Delta Q_\mathrm{J}}{T_\mathrm{J}}\,,$$

and we already know that we want to minimize ΔQ_J. So we can see that we want to consider equality in the equation (14) (which is easier to realize if we move the other terms without ΔQ_J to the right-hand side) if we are interested in maximizing the amount of work. Next, note that $\Delta S_W = 0$ helps us minimize ΔQ_J, meaning that we do the work reversibly/adiabatically. Finally, we indeed get a condition on the reversible process when $\Delta S = 0$

$$\Delta S_\mathrm{J} = \frac{\Delta Q_\mathrm{J}}{T_\mathrm{J}} = -\Delta S_\mathrm{geo} = -\int \frac{\mathrm{d}Q}{T} = -\int_{T_1}^{T_\mathrm{J}} c\rho V_1 \frac{\mathrm{d}T}{T}\,,$$

from where we have

$$\Delta Q_\mathrm{J} = -c\rho V_1 T_\mathrm{J} \ln\left(\frac{T_\mathrm{J}}{T_1}\right)\,.$$

For the work from (13), it holds

$$W = c\rho V_1 \left[T_1 - T_\mathrm{J} + T_\mathrm{J} \ln\left(\frac{T_\mathrm{J}}{T_1}\right)\right] ,$$

knowing that when pumping water, we are doing work against the gravitational field $W = \rho \Delta V g h$ and for the maximum amount of water V_max we have

$$V_\mathrm{max} = \frac{c V_1 \left[T_1 - T_\mathrm{J} + T_\mathrm{J} \ln\left(\frac{T_\mathrm{J}}{T_1}\right)\right]}{gh} .$$

This is for the values from the input $V_\mathrm{max} = 16\,277\,\mathrm{m^3}$. No matter what we do, no machine pumps more water using geothermal water energy. However, this is a remarkably large amount of water. Consider that to pump $16\,277\,\mathrm{m^3}$ from a depth of 50 meters; we only need $200\,\mathrm{m^3}$ of geothermally heated water and a bath. The reason for this is, of course, the high heat capacity of the water. Finally, let us note that while this is a lot of water, it is still much less than we might naively expect from $V = cV_1/gh = 136\,481\,\mathrm{m^3}$.

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Problem 57 ... a beer problem 9 points

Drinking beer is not as simple as it seems. You must pick up your half-pint and tilt it so much that it starts pouring into your mouth. For this, however, work must be done. Consider the half-liter weight of $m = 360\,\mathrm{g}$, its cylindrical shape of radius $r = 3.5\,\mathrm{cm}$ and height $h = 15\,\mathrm{cm}$ and density of beer $\rho = 1\,030\,\mathrm{kg{\cdot}m^{-3}}$. Determine the work done if you lift a half-liter filled with 350 ml of beer to drink. Your mouth is 40 cm above the table.

Jarda was thinking about how much he has to pay Viktor again...

To drink, we must do the work necessary to lift the tankard in a gravitational field, e.g. to increase its potential energy. At first, we will calculate the angle of the tilt of a half-liter so that a beer will start pouring out of it. The fluid will take shape consisting of a cylinder of height x and a shape that gets created by an oblique cut from one end of the first base to the other end of the other base. We will designate the height of this shape as y. Then, the overall volume of these two bodies is equal to the volume of the beer in the glass V

$$V = \pi r^2 x + \frac{1}{2}\pi r^2 y .$$

For the beer to pour out, a condition of $x + y = h$ must be fulfilled. The tilt angle can also be expressed as $\tan\alpha = y/2r$, where α is the angle by which the height of the half-liter is diverted from the vertical. From these three equations, we get the wanted angle as

$$\tan\alpha = \frac{\pi r^2 h - V}{\pi r^3} \quad \Rightarrow \quad \alpha = 59.3\,^\circ .$$

At this angle arises a situation where there is enough beer in the tankard that its whole base is still completely covered with beer. The situation would change at an angle of $\arctan(h/2r) = 65.0\,^\circ$.

From the relations before we can express numerically $y = 2r\tan\alpha = 11.81\,\mathrm{cm}$ and $x = 3.19\,\mathrm{cm}$.

Next, we will find at what height above the base of an empty half-liter is the center of mass. Since it is made out of a homogeneous material with a consistent width of walls, we can designate the areal density of the walls and the base as σ. The center of mass of the base is at a height of zero above the bottom, the center of mass of the walls is at a height of $h/2$, the overall height of the center of mass of the half-liter is therefore

$$y_\mathrm{k} = \frac{\frac{h}{2} 2\pi r h\sigma}{2\pi r h\sigma + \pi r^2\sigma} = \frac{h^2}{2h+r} = 6.72\,\mathrm{cm}\,.$$

Calculating the center of mass of the space, which is taken by the beer, will be more difficult. We get it from the knowledge of the location of the center of mass of the cylinder, which lies $x/2$ above the base, and from the location of the center of mass of the second part of the shape the beer makes. We can calculate it using integration as

$$y_y = \frac{1}{\rho\left(\frac{1}{2}\pi r^2 y\right)}\int_{-r}^{r} 2\sqrt{r^2-u^2}\,(u+r)\,\frac{y}{2r}\rho\,(u+r)\,\frac{y}{4r}\,\mathrm{d}u \;=\frac{1}{(\pi r^3)}\frac{y}{2r}\frac{5\pi r^4}{8} = \frac{5y}{16} = 3.69\,\mathrm{cm}\,.$$

We will get the height of the center of mass of the beer above the base simply as

$$y_\mathrm{p} = \frac{1}{V}\left(\frac{x}{2}\pi r^2 x + \left(\frac{5y}{16}+x\right)\frac{1}{2}\pi r^2 y\right) = 5.03\,\mathrm{cm}\,.$$

The center of mass of the beer, however, will not be on the axis of symmetry of the cylinder but will be shifted in the direction closer to the ground. Analogically, we will calculate its distance from the axis of symmetry. The center of mass of the cylindrical part is on the axis of symmetry, and the center of mass of the other part is at a distance of

$$x_y = \frac{1}{\rho\left(\frac{1}{2}\pi r^2 y\right)}\int_{-r}^{r} 2\sqrt{r^2-u^2}\,(u+r)\,\frac{y}{2r}\rho u\;\mathrm{d}u \;= \frac{2}{\pi r^3}\frac{1}{8}\pi r^4 = \frac{1}{4}r = 0.875\,\mathrm{cm}\,.$$

The center of mass of the beer is, therefore, shifted by

$$x_\mathrm{p} = \frac{1}{V}\frac{1}{2}\pi r^2 y\frac{1}{4}r = 0.568\,\mathrm{cm}$$

from the axis of symmetry of the half-liter.

Let us designate $H = 40\,\mathrm{cm}$ as the height of the mouth above the table. The center of the base will be located at a height of

$$v_\mathrm{s} = H - h\cos\alpha + r\sin\alpha = 35.36\,\mathrm{cm}\,.$$

The center of mass of the half-liter itself is then higher by $y_\mathrm{k}\cos\alpha$. The center of mass of the beer is higher by $y_\mathrm{p}\cos\alpha - x_\mathrm{p}\sin\alpha$. The potential energy of the tilted beer and the half-liter is, relative to the ground,

$$E = g\left(m\left(v_\mathrm{s} + y_\mathrm{k}\cos\alpha\right) + V\;\rho\left(v_\mathrm{s} + y_\mathrm{p}\cos\alpha - x_\mathrm{p}\sin\alpha\right)\right) = 2.694\,\mathrm{J}\,.$$

From this value, it is necessary to subtract the potential energy of the center of mass of the half-liter when it is sitting on the table. That is equal to

$$E_0 = gm\frac{h^2}{2h+r} + gV\rho\frac{V}{2\pi r^2} = 0.398\,\mathrm{J}\,.$$

Therefore, the result is a work of 2 296 mJ.

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Problem K.1 ... the pendulum swings back... 3 points

In certain buildings, one might encounter Foucault's pendulum, which has a very long suspension. Jarda once encountered one, wanting to estimate the height of the ceiling from which the pendulum was suspended. He measured the period of oscillation as 14.2 s*, and when the pendulum was in its lowest position, it was* 70 cm *above the floor. Determine the height of the ceiling.* *Jarda has been in the Panthéon in Paris.*

In this solution, we consider the pendulum to be a mathematical one. The well-known relation that connects the length of the suspension L, gravity of Earth g and the period of oscillation T is given by the formula:

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \Rightarrow \quad L = \frac{T^2}{4\pi^2}g = 50.1\,\mathrm{m}\,.$$

In order to find the height of the point of suspension above the ground, it is neccessary to add the lowest height of the pendulum above the floor h to the length of the suspension. This yields the final result:

$$H = L + h = \frac{T^2}{4\pi^2}g + h = 50.8\,\mathrm{m}\,.$$

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Problem K.2 ... ...and forth... 3 points

To be able to use the approximation of a mathematical pendulum, we need a point of mass on an massless suspension. Let's consider a string made of steel with a diameter of 1.4 mm *on which a weight is suspended. For good accuracy, we require it to be* 80 *times more massive than the string on which it is suspended. If the tensile strength of the used steel is* $520\,\mathrm{N{\cdot}mm^{-2}}$*, what can be the maximum length of the suspension at rest? The density of the used steel is* $7\,900\,\mathrm{kg{\cdot}m^{-3}}$*.* *Such a thin and long rope, and yet it would still hold Jarda.*

The highest tension in the suspension will be at the point of suspension because it bears the mass of the weight and also of the rest of the suspension. The mass of the suspension will be

$$m_\mathrm{c} = \frac{\pi d^2}{4}\rho L\,,$$

where d is its diameter, ρ is the density of steel and L is its length. For the mass of the weight, it applies $m_\mathrm{w} = k m_\mathrm{c}$, where $k = 80$.

The force acting on the suspension at the suspension point is

$$F = \left(m_\mathrm{c} + m_\mathrm{w}\right)g = m_\mathrm{c}\left(1 + k\right)g = \frac{\pi d^2}{4}\rho L\left(1 + k\right)g\,.$$

If we divide this force by the cross-sectional area of the suspension, we get the stress in the material. The suspension must not break, so the following holds

$$\sigma = \frac{F}{S} = \rho L \left(1 + k\right) g \quad \Rightarrow \quad L = \frac{\sigma}{\rho g \left(1 + k\right)} \doteq 83\,\mathrm{m}\,,$$

where σ is the tensile strength.

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Problem K.3 ... ...and back... 4 points

The horizontal displacement of a pendulum has lowered from $1.0\,\mathrm{m}$ *to* $0.9\,\mathrm{m}$ *in ten minutes. Determine the average work the resistance forces do during a single pendulum period. Again, assume a mathematical pendulum with a* $47\,\mathrm{kg}$ *weight and a period of oscillation of* $14.2\,\mathrm{s}$*.*

Jirka thought the statement of this problem was confusing.

It is important that the period of the oscillation is not dependent on the displacement of the pendulum. In time $t = 10\,\mathrm{min}$ it has made

$$N = \frac{t}{T} = 42.3\,.$$

Oscillations. Further, we know that the length of the pendulum can be calculated from the period of the oscillation as

$$L = \frac{T^2 g}{4\pi^2} = 50.2\,\mathrm{m}\,.$$

The pendulum has lost some energy, which we will calculate from the difference of potential energies. From the knowledge of the displacement $x = L\sin\varphi$, where φ is the angle of deviation from the vertical, we can calculate the decrease of potential energy of the pendulum as

$$\Delta E = mgL\left(\cos\varphi_\mathrm{f} - \cos\varphi_\mathrm{i}\right)\,.$$

We will designate index i as the initial angular displacement and index f as the final. Because the initial angle holds the equation $\varphi_\mathrm{i} = \arctan\left(\frac{x_\mathrm{i}}{L}\right) \doteq 0.020\,\mathrm{rad} \ll 1$, we can with a good precision use approximations $\tan\varphi \approx \sin\varphi \approx \varphi$ a $\cos\varphi \approx 1 - \frac{\varphi^2}{2}$. Therefore, the difference of the potential energies can be written as

$$\Delta E = \frac{mgL}{2}\left(\varphi_\mathrm{i}^2 - \varphi_\mathrm{f}^2\right) = \frac{mg}{2L}\left(x_\mathrm{i}^2 - x_\mathrm{f}^2\right)\,.$$

This energy is equal to the work made by the resistance forces. On average, the pendulum loses during one period an energy

$$P = \frac{\Delta E}{N} = \frac{mg\left(x_\mathrm{i}^2 - x_\mathrm{f}^2\right)T}{2Lt}\,.$$

From the knowledge of the period of the oscillation, we will substitute for the length of the suspension $L = \frac{T^2 g}{4\pi^2}$, and we get

$$P = \frac{\Delta E}{N} = 2\pi^2\frac{m\left(x_\mathrm{i}^2 - x_\mathrm{f}^2\right)}{Tt} \doteq 21\,\mathrm{mJ}\,.$$

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Problem K.4 ... ...and forth, over and over again 5 points

The estimate did not seem accurate enough to Jarda, so he decided not to consider the pendulum as a simple mathematical pendulum but went on to include the moments of inertia of its parts in his calculations. He found that the suspension consisted of 50.0 m *long string made from steel with a diameter of* 1.40 mm *and that at the end of the suspension, there was a ball of radius* 10.0 cm *with mass* 17.0 kg*. What is the ratio of the period of such a pendulum to the period of a mathematical pendulum of length* 50.1 m*? The density of steel is* $7.90\,\mathrm{g\cdot cm^{-3}}$*.*

Jarda needed a physical pendulum problem also for this year's Hurry-up.

We calculate the period using the formula for the physical pendulum, which is

$$T = 2\pi\sqrt{\frac{J}{mgx}}\,,$$

where J is the moment of inertia of the body with respect to the rotational axis, m is its mass, g is the gravitational acceleration, and x is the distance of the center of gravity of the body from the rotational axis.

The mass of the string is $m_d = \rho V = \rho\frac{\pi d^2}{4}l$, where the density of the steel is equal to $\rho = 7\,900\,\mathrm{kg\cdot m^{-3}}$. We denote the volume of the string as V and then express it in terms of diameter d and length l. The distance of the center of gravity from the rotational axis is thus

$$x = \frac{m_\mathrm{d}\frac{l}{2} + M\left(l+R\right)}{m_\mathrm{d} + M} = \frac{\rho\frac{\pi d^2}{8}l^2 + M\left(l+R\right)}{m_\mathrm{d} + M}\,,$$

where M is the mass of the ball at the end and R is its radius.

The moment of inertia of the whole body can be obtained by summing the moments of inertia of its individual parts. The suspension can be considered as a thin, rigid rod that rotates around one of its ends, which corresponds to the moment of inertia

$$J_\mathrm{d} = \frac{1}{3}m_\mathrm{d}l^3 = \frac{1}{12}\rho\pi d^2 l^3\,.$$

The moment of inertia of the ball with respect to the axis passing through its center of gravity is $\frac{2}{5}MR^2$, but in this situation, we have to shift the axis by $M\left(l+R\right)^2$ according to parallel axis theorem, and we get the total moment of inertia of the whole pendulum as

$$J = \frac{1}{12}\rho\pi d^2 l^3 + \frac{2}{5}MR^2 + M\left(l+R\right)^2\,.$$

By plugging in the initial relation and comparing it with the period of a mathematical pendulum of length $(l+R)$ we get

$$\frac{T}{T_\mathrm{mat}} = \sqrt{\frac{J}{\left(m_\mathrm{d}+M\right)\left(l+R\right)x}} = \sqrt{\frac{1 + \frac{\frac{1}{12}\rho\pi d^2 l^3 + \frac{2}{5}MR^2}{M(l+R)^2}}{1 + \frac{\rho\frac{\pi d^2}{8}l^2}{M(l+R)}}} \doteq 0.997\,.$$

The difference is so small that even for such a long period of oscillation, it would not be easy to measure using a stopwatch.

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Problem T.1 ... inflating a balloon

4 points

When we are inflating a balloon with a large enough radius, we can accurately assume the balloon's elastic potential energy to be proportional to its surface area. Jirka found such a balloon with a radius of 5 cm and measured the air pressure inside it to be equal to 107 kPa. However, the balloon was not very large, so he decided to inflate it to a radius of 15 cm. How many moles of air did he need to blow into the balloon? Assume a spherical balloon surrounded by room air with a temperature $T = 20\,^\circ\mathrm{C}$. *Optics tutorial was too boring for Jirka.*

We want to use the ideal gas law to compute the amount of air we have to blow:

$$pV = nRT\,.$$

In this equation, we know the temperature $T = 20\,^\circ\mathrm{C}$ and the volume $V_2 = \frac{4}{3}\pi r_2^3$, where $r_2 = 15\,\mathrm{cm}$ (spherical balloon). We have to relate the pressure inside the balloon to its size.

We already know that to increase the area of the balloon by $\mathrm{d}S$ (consider $\mathrm{d}S$ to be an infinitesimally small area, although the relation would hold even for a finite ΔS), we must do work $\mathrm{d}W$ against the balloon's forces such that

$$\mathrm{d}W = \frac{A}{2}\cdot \mathrm{d}S\,.$$

where we denoted the proportionality constant $\frac{A}{2}$ on behalf of consistency with other problems in this Hurry up series.

The balloon has a spherical shape, so the resultant force acts toward the center (the balloon is trying to shrink). During an enlargement by some small radius $\mathrm{d}r$, the work is done

$$\mathrm{d}W = F\cdot \mathrm{d}r = p\cdot S\cdot \mathrm{d}r = p\cdot 4\pi r^2\,\mathrm{d}r\,,$$

where p is the pressure exerted by the balloon at the radius r. We took advantage of the fact that $\mathrm{d}r$ is small, then during the increase of r by Δr we can consider the force F and the area S constant (i.e. we omit the terms $O(\mathrm{d}r^2)$). Similarly, we could proceed by reasoning that the pressure in the balloon does the same work in inflating it as the pressure of the gas does in expanding it. When the gas expands by a volume $\mathrm{d}V$, it does work $\mathrm{d}W = p\,\mathrm{d}V$, where we also have $\mathrm{d}W = p\cdot 4\pi r^2\,\mathrm{d}r$.

We do know as well that the work is proportional to the area $\mathrm{d}S = 8\pi r^2\,\mathrm{d}r$ and to sum up, we can write

$$p\cdot 4\pi r^2\,\mathrm{d}r = \frac{A}{2}\cdot 8\pi r^2\,\mathrm{d}r\,,$$

where we find the dependance of the pressure on the radius

$$p = \frac{A}{r}\,.$$

Please note that this is not yet the total pressure in the balloon. The air is also subjected to the atmospheric pressure $p_\mathrm{a} = 101\,325\,\mathrm{Pa}$. Finally, the total pressure is $p + p_\mathrm{a}$.

Given the initial pressure p_1 and the radius r_1, we determine the constant A as $A = (p_1 - p_\mathrm{a})r_1$. Then we obtain the number of moles of the air that Jirka must blow into the balloon

$$\Delta n = \frac{p_2V_2}{RT} - \frac{p_1V_1}{RT} \qquad \Rightarrow \qquad \Delta n = \frac{4\pi}{3RT}\left[\left(p_\mathrm{a} + (p_1 - p_\mathrm{a})\frac{r_1}{r_2}\right)r_2^3 - p_1 r_1^3\right]\,.$$

Substituting the numbers, we get $\Delta n = 0.576\,\mathrm{mol}$, which translates to around 13 liters of air at standard pressure. Humans are able to inhale approximately 3 liters of air, so Jirka needs about 5 breaths to inflate the balloon.

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Problem T.2 ... we are cooling down the balloon 5 points

Jirka was already satisfied with the size of his balloon and took it for a walk. He left the heated room at $20\,\mathrm{^\circ C}$ *wearing only a T-shirt and found out that he was really cold because the outside temperature was only* $3\,\mathrm{^\circ C}$*. What is the new radius of the balloon after it shrank during the walk if it initially had a radius of* $r_1 = 15\,\mathrm{cm}$*? Do not forget that in the previous problem, you derived a relationship between the overpressure in the balloon and its radius (assuming the balloon has a spherical shape)*

$$\Delta p = \frac{A}{r}\,,$$

where $A = 300\,\mathrm{Pa{\cdot}m}$*.* *Jirka attends secret night meetings.*

We start from the equation of state for an ideal gas. Assuming that the air inside the balloon does not escape during the walk, we have

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}\,,$$

where $T_1 = 20\,\mathrm{^\circ C}$, $T_2 = 3\,\mathrm{^\circ C}$. The balloon is spherical, so its volume is

$$V = \frac{4}{3}\pi r^3\,.$$

The pressure of the air in the balloon is equal to the sum of the atmospheric pressure p_a and the overpressure in the balloon. Therefore, we have

$$p = p_\mathrm{a} + \frac{A}{r}\,.$$

Now, we substitute all the information into the equation of state, and after some adjustments, we get

$$\frac{T_2}{T_1}\left(p_\mathrm{a} + \frac{A}{r_1}\right) r_1^3 = A r_2^2 + p_\mathrm{a} r_2^3\,.$$

Since it is a cubic equation, we will settle for a numerical solution. The only real root is $r_2 \doteq 14.7\,\mathrm{cm}$. Notice that in this case, the change in pressure in the balloon influences the result very little. If we considered a constant pressure inside the balloon, we would obtain a result that differs on the order of tenths of a percent. Therefore, we get

$$r_2 = r_1 \sqrt[3]{\frac{T_2}{T_1}} \doteq 14.7\,\mathrm{cm}\,.$$

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Problem T.3 ... our balloon has flown away 6 points

Viktor had gifted Jirka a little present for his drive during proofreading – a helium balloon with radius $r_0 = 11\,\mathrm{cm}$. Jirka weighted down the balloon in a way that the total weight of the material was $M = 5.2\,\mathrm{g}$ and hoped that it was enough for the balloon not to fly away. Unfortunately, he was mistaken, and the balloon started to rise. If we consider the same dependency between the radius and the pressure inside the balloon as in the problems before, i.e.

$$\Delta p = \frac{A}{r}\,,$$

where $A = 300\,\mathrm{Pa{\cdot}m}$, determine to what height the balloon will rise in an isothermic atmosphere with a temperature $T = 20\,\mathrm{^\circ C}$, provided that it won't pop. The molar mass of helium is $4.003\,\mathrm{g{\cdot}mol^{-1}}$.

Hint: *In an isothermic atmosphere both the pressure and the density decrease exponentially.*

Actually, Jarda invited Jirka to have a beer.

An isothermic atmosphere has a constant temperature everywhere and for its pressure p_a and density ρ_a the following holds

$$p_\mathrm{a} = p_\mathrm{a0} \exp\left(-\frac{h}{h_0}\right)\,, \rho_\mathrm{a} = \rho_\mathrm{a0} \exp\left(-\frac{h}{h_0}\right)\,,$$

where p_a0, ρ_a0 are the pressure and the density at the ground level and $h_0 = \frac{RT}{gM_\mathrm{a}} = 8600\,\mathrm{m}$ is the height ($M_\mathrm{a} = 28.96\,\mathrm{g\cdot mol^{-1}}$) where both pressure and density decrease to 1/e of the values at the ground level. Furthermore, the pressure and the density are in a relation

$$\frac{p_\mathrm{a}}{\rho_\mathrm{a}} = \frac{RT}{M_\mathrm{m}}\,,$$

where T is the temperature of the gas and M_m is its molar mass.

Another relation that will accompany us for the rest of the problem is the relation for pressure p_i of the helium inside the balloon

$$p_\mathrm{i} - p_\mathrm{a} = \frac{A}{r}\,.$$

Under normal circumstances, we can determine the density of the helium simply as

$$\rho_\mathrm{He0} = \frac{p_\mathrm{a0} M_\mathrm{He}}{RT} = 0.1664\,\mathrm{kg{\cdot}m^{-3}}\,,$$

where $M_\mathrm{He} = 4.003\,\mathrm{g\cdot mol^{-1}}$ is the molar mass of helium. The density of the helium inside the balloon will be somewhat greater because there is a pressure larger by $\frac{A}{r_0}$; therefore

$$\rho_\mathrm{He} = \rho_\mathrm{He0} \frac{p_\mathrm{a0} + A/r_0}{p_\mathrm{a0}} = 0.1709\,\mathrm{kg{\cdot}m^{-3}}\,.$$

For the balloon to float at a certain height, its gravitational force has to be equal to its upthrust force

$$mg = V \rho_\mathrm{a} g \quad \Rightarrow \quad m = \frac{4}{3}\pi r^3 \rho_\mathrm{a}\,.$$

We know that at the ground level, the upthrust force was greater than the gravitational force, which is why we have weighted the balloon down.

We can find the weight of the balloon and the gas inside as

$$m = \frac{4}{3}\pi r_0^3 \rho_{\mathrm{He}} + M = 6.153\,\mathrm{g}\,.$$

The last important equation is the equation of state of an ideal gas in the balloon, according to which

$$p_{\mathrm{i}} \frac{4}{3}\pi r^3 = nRT\,.$$

From this equation, we will simply substitute in the equation for pressure and express the atmospherical pressure depending on the radius as

$$\frac{3nRT}{4\pi r^3} - \frac{A}{r} = p_{\mathrm{a0}} \exp\left(-\frac{h}{h_0}\right)\,.$$

We plug the exponential into the balance of forces equation

$$\exp\left(-\frac{h}{h_0}\right) = \frac{\rho_{\mathrm{a}}}{\rho_{\mathrm{a0}}} = \frac{m}{\frac{4}{3}\pi r^3 \rho_{\mathrm{a0}}}\,,$$

and we get the equation for r in the form

$$m = \frac{4}{3}\pi r^3 \frac{\rho_{\mathrm{a0}}}{p_{\mathrm{a0}}}\left(\frac{3nRT}{4\pi r^3} - \frac{A}{r}\right) \quad \Rightarrow \quad r = \sqrt{\frac{3}{4\pi A}\left(nRT - \frac{m p_{\mathrm{a0}}}{\rho_{\mathrm{a0}}}\right)}\,.$$

Because the temperature and the molar amount of helium in the ballon are constant, for the product nRT, we can write

$$nRT = p_{\mathrm{i0}} \frac{4}{3}\pi r_0^3 = \left(\frac{A}{r_0} + p_{\mathrm{a0}}\right)\frac{4}{3}\pi r_0^3$$

and substitute it into the preceding equation.

Finally, we substitute the radius into the forces equation, and after some additional adjustments and substitutions, we can express the height h as

$$h = h_0 \ln\left(\frac{\rho_{\mathrm{a0}}}{\rho_{\mathrm{a}}}\right) = h_0 \ln\left(\frac{4\pi r^3 \rho_{a0}}{3m}\right) = h_0 \ln\left(\frac{4\pi\left(\sqrt{r_0^2 + \frac{r_0^3 p_{\mathrm{a0}}}{A} - \frac{3m p_{\mathrm{a0}}}{4\pi A \rho_{\mathrm{a0}}}}\right)^3 \rho_{a0}}{3m}\right) \doteq 19\,\mathrm{km}\,.$$

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Problem T.4 ... connecting balloons

6 points

Because Jirka's helium balloon had flown away, he sadly had to ask Viktor for another one. He got two, but he had to inflate them himself. Jirka inflated one to a radius of $r_i = 15.0\,\mathrm{cm}$, then took a short, narrow tube and connected it to the other so that no air escaped. To his great surprise, once the balance was established, one balloon had a larger radius than the other. What was the ratio of the radius of the larger balloon to the radius of the smaller one? Now, suppose that the pressure difference between the balloon and the surrounding area depends on its radius as

$$\Delta p = \frac{A}{r}\left[1 - \left(\frac{r_0}{r}\right)^6\right],$$

where $A = 300\,\mathrm{Pa{\cdot}m}$ and $r_0 = 3.00\,\mathrm{cm}$. The experiment was conducted at a $20\,^{\circ}\mathrm{C}$. Assume the second balloon has radius r_0 before attaching it to the tube.

Jarda saw an interestig experiment at the lecture.

The situation stabilizes once the pressures between the balloons are equal while the total amount of substance of gas in the system is preserved. Knowing the relationship for pressure as a function of radius, we can use the equation of state to determine the amount of substance of gas in the system as

$$n_1 = \frac{1}{RT}p\frac{4}{3}\pi r_i^3 = \frac{1}{RT}\left\{\frac{A}{r_i}\left[1 - \left(\frac{r_0}{r_i}\right)^6\right] + p_\mathrm{a}\right\}\frac{4}{3}\pi r_i^3 \doteq 0.599\,\mathrm{mol},$$

$$n_2 = \frac{1}{RT}p_\mathrm{a}\frac{4}{3}\pi r_0^3 \doteq 0.005\,\mathrm{mol}.$$

Let r_1 denote the radius of the original of the connected balloons and r_2 the radius of the second. With equal pressures, the following holds

$$\frac{1}{r_1}\left[1 - \left(\frac{r_0}{r_1}\right)^6\right] = \frac{1}{r_2}\left[1 - \left(\frac{r_0}{r_2}\right)^6\right].$$

We can always use one radius from this equation to calculate the second radius.

At the same time, the equation of state for the whole system must be of the form

$$p\frac{4\pi}{3}\left(r_1^3 + r_2^3\right) = \left\{\frac{A}{r_1}\left[1 - \left(\frac{r_0}{r_1}\right)^6\right] + p_\mathrm{a}\right\}\frac{4\pi}{3}\left(r_1^3 + r_2^3\right) = (n_1 + n_2)RT \doteq 1472.18\,\mathrm{J},$$

We know the value of the right-hand side of the equation; the only variable on the left-hand side is r_1. So we will vary r_1 until we get the equality between the two sides to the desired precision. For example, we can use software like Wolfram Mathematica or a suitable graphing calculator like GeoGebra. We plot the dependence of the overpressure on the radius, place one point on it, and find another point that has the same overpressure and is, therefore, at the intersection of the overpressure curve with the parallel x axis, so we know r_1 and r_2. We find that for the right side to equal nRT, we need $r_1 = 14.994\,\mathrm{cm}$ and $r_2 = 3.119\,\mathrm{cm}$. Their ratio and, therefore, the solution to our problem is 4.81.

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Problem M.1 ... tram

3 points

What is the maximum angle at which a tram can travel downhill and still be able to stop? The coefficient of shear friction of the wheels and rails is $f = 0.15$.

David rode to the lecture suspiciously downhill.

Suppose the tram is on an inclined plane at an angle α with the ground. In that case, three forces act on it: the downward gravitational force F, the normal force $N = F\cos\alpha$, which is perpendicular to the inclined plane, and the frictional force F_f acting against the direction of motion. The maximum possible value of the friction force is

$$F_{\mathrm{f,\,max}} = fN = fF\cos\alpha\,,$$

where $f = 0.15$ is the coefficient of shear friction between the tram wheels and the rails. This is also the maximum braking force the tram can exert.

In the downhill direction, the tangential component of the gravitational force

$$F_{\|} = F\sin\alpha\,,$$

is trying to get the tram moving. The tram can only stop if its braking force is greater than the accelerating tangential component of the gravitational force. We get an inequality between these two forces

$$\begin{aligned}
F_{\mathrm{f,\,max}} &> F_{\|}\\
fF\cos\alpha &> F\sin\alpha\\
\tan\alpha &< f\\
\alpha &< \arctan f\\
\alpha &< 8.53^\circ \doteq 8.5^\circ.
\end{aligned}$$

The tram can only stop safely on slopes with an incline less than 8.5°.

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Problem M.2 ... the tram reloaded

3 points

The tram travels at $v = 27\,\mathrm{km{\cdot}h^{-1}}$ and its maximum deceleration is $a = 2.1\,\mathrm{m{\cdot}s^{-2}}$. What is the minimum distance required for the tram to come to a complete stop upon Davis's signaling at the stop? The reaction time of the driver to the David's wave is $t' = 0.3\,\mathrm{s}$.

David studied the T3 tram manual for far too long

First, we will calculate the distance the tram travels before the driver reacts to David as

$$s_1 = vt' = 2.25\,\mathrm{m}\,.$$

We will then calculate how long it takes for the tram to stop

$$v = at \quad\Rightarrow\quad t = \frac{v}{a} \doteq 3.57\,\mathrm{s}\,.$$

After that, we will determine the stopping distance using the well-known formula

$$s_2 = vt - \frac{1}{2}at^2 = \frac{1}{2}at^2 \doteq 13.39\,\mathrm{m}.$$

Finally, we will add the two distances together

$$s = s_1 + s_2 \doteq 16\,\mathrm{m}\,.$$

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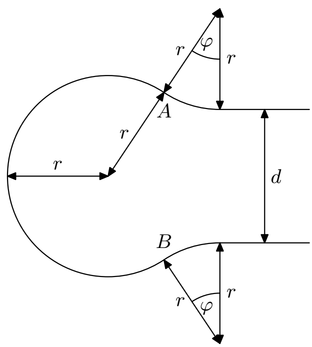
Problem M.3 ... parallel tracks

4 points

Consider two parallel tram tracks at a distance of $d = 11\,\mathrm{m}$. We want to build a system of arcs between them with a radius r, as shown in the right image. Find r such that a tram, capable of withstanding a centripetal acceleration of $a = 0.85\,\mathrm{m{\cdot}s^{-2}}$ in turn, passes through it in the shortest possible time. The velocity of the tram remains constant.

Adam would like to ride the Snowpiercer.



Obviously, only $r \geq d/2$ makes sense. Let's choose any r that satisfies this condition and calculate the total length of the arcs $s(r)$ and the maximum possible speed of the locomotive $v(r)$.

Speed is much simpler, so let's start with it. Because $a \leq v^2/r$ it holds

$$v = \sqrt{ar}\,.$$

To calculate the function $s(r)$, let's first determine the central angle corresponding to the middle arc. For that, we have $\alpha = \pi + 2\varphi$. Subsequently, we express distances A and B in two ways. Using the large arc as

$$2r \sin\left(\frac{\pi}{2} + \varphi\right) = 2r\cos\varphi$$

and using the small arcs as

$$d + 2r(1 - \cos\varphi).$$

We will compare these two expressions and obtain the relation $\varphi = \arccos\left[(d + 2r)/4r\right]$. Finally, it is sufficient to add up the individual arcs and obtain

$$s(t) = \pi r + 4r \arccos\left(\frac{d + 2r}{4r}\right).$$

From the known functions $s(r)$ and $v(r)$, we will express time as a function r

$$t(r) = \frac{s(r)}{v(r)} = \pi\sqrt{\frac{r}{a}} + 4\sqrt{\frac{r}{a}}\arccos\left(\frac{d + 2r}{4r}\right) \leq \pi\sqrt{\frac{r}{a}}\,.$$

Function $f(r) = \pi\sqrt{r/a}$ is increasing, and the tram will pass through the arc fastest when $r = d/2 = 5.5\,\mathrm{m}$.

Note: In fact, it was not necessary to express the angle φ (it is sufficient that it is non-negative), but it is an interesting geometric problem.

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Problem M.4 ... turboflies' troubles

5 points

Two trams at a distance of $s = 400\,\mathrm{m}$ are moving against each other, first with a velocity of $v_1 = 30\,\mathrm{km{\cdot}h^{-1}}$, second with a velocity of $v_2 = 35\,\mathrm{km{\cdot}h^{-1}}$. From the first one, a fly takes off with a velocity of $v_3 = 80\,\mathrm{km{\cdot}h^{-1}}$ and flies to the other tram, where it bounces off the windshield, flies back, and so on, as long as it manages to outfly the trams. The windshields are sticky, so with each bounce, its speed gets lowered to a q-factor of the velocity before the bounce, where $q = 0.97$. Assuming the bounce is instantaneous, what distance does the fly cover before the trams collide?

Marek has seen a fly in a tram.

After time t since the moment the fly takes off, the trams are at a distance of $s - (v_1 + v_2)t$. We will define the time of the i-th collision with the trams as t_i (and assign $t_0 = 0\,\mathrm{s}$). Next, we will designate $v_1 = 30\,\mathrm{km{\cdot}h^{-1}}$ as the velocity of the first tram and v_2 as the velocity of the second one, while the initial velocity of the fly will be designated as v_3. Then, we can write for even and odd collisions:

$$s - (v_1 + v_2)t_{2n} = \left(v_2 + v_3 \cdot q^{2n}\right)\left(t_{2n+1} - t_{2n}\right) ,$$
$$s - (v_1 + v_2)t_{2n+1} = \left(v_1 + v_3 \cdot q^{2n+1}\right)\left(t_{2n+2} - t_{2n+1}\right) ,$$

because the fly and the oncoming tram have to cover the distance between the trams in the same time. Upon rearranging, we get iterative relations

$$t_{2n+1} = \frac{s - \left(v_1 - v_3 \cdot q^{2n}\right) t_{2n}}{v_2 + v_3 \cdot q^{2n}} ,$$
$$t_{2n+2} = \frac{s - \left(v_2 - v_3 \cdot q^{2n+1}\right) t_{2n+1}}{v_1 + v_3 \cdot q^{2n+1}} .$$

We can notice that if the fly did not slow down, a formally infinite number of collisions with the trams would happen. However, the fly is slowing down, thus its velocity will be smaller at one point than the velocity of one of the trams, and it will only ride the windshield. This happens for least such k that one of the following inequalities will hold

$$v_3 q^{2k} \leq v_1 , \quad v_3 q^{2k+1} \leq v_2 .$$

This allows us to determine the number of collisions with trams N as $2k$ if the first inequality holds or as $2k + 1$ if the second inequality is met. k can be determined using, for example, any spreadsheet, such as Excel. Once we determine it, we will calculate the total path of the fly as

$$s = \sum_{n=1}^{N} v_3 q^{n-1} (t_n - t_{n-1}) ,$$

We could have observed that after about the tenth bounce, the trams are so close that the covered distances don't change significantly, so performing a summation until index $N = 10$ would be sufficient.

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